

# Finite temperature fermion condensate, charge and current densities in a $(2 + 1)$ -dimensional conical space

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**Abstract** We evaluate the fermion condensate and the expectation values of the charge and current densities for a massive fermionic field in  $(2 + 1)$ -dimensional conical spacetime with a magnetic flux located at the cone apex. The consideration is done for both irreducible representations of the Clifford algebra. The expectation values are decomposed into the vacuum expectation values and contributions coming from particles and antiparticles. All these contributions are periodic functions of the magnetic flux with the period equal to the flux quantum. Related to the non-invariance of the model under the parity and time-reversal transformations, the fermion condensate and the charge density have indefinite parity with respect to the change of the signs of the magnetic flux and chemical potential. The expectation value of the radial current density vanishes. The azimuthal current density is the same for both the irreducible representations of the Clifford algebra. It is an odd function of the magnetic flux and an even function of the chemical potential. The behavior of the expectation values in various asymptotic regions of the parameters are discussed in detail. In particular, we show that for points near the cone apex the vacuum parts dominate. For a massless field with zero chemical potential the fermion condensate and charge density vanish. Simple expressions are derived for the part in the total charge induced by the planar angle deficit and magnetic flux. Combining the results for separate irreducible representations, we also consider the fermion condensate, charge and current densities in parity and time-reversal symmetric models. Possible applications to graphitic nanocones are discussed.

## 1 Introduction

Fermionic field theoretical models in three-dimensional spacetime rise in a number of physical problems. In particular, the long-wavelength description of a variety of planar condensed matter systems can be formulated in terms of the Dirac-like theory. The examples include models of high-temperature superconductivity, graphene, d-density-wave states and topological insulators (for reviews see [1–5]). The long-wavelength dynamics, equivalent to relativistic Dirac fermions with a controllable mass, is exhibited by ultracold fermionic atoms in an optical lattice [6]. Another motivation comes from the connection of three-dimensional models to high temperature behavior in 4-dimensional field theories [7]. Field theories in three dimensions also provide simple models in particle physics. Because of one dimension less they are easier to handle.

Three-dimensional theories exhibit a number of interesting features, such as flavour symmetry breaking, parity violation, fractionalization of quantum numbers, that make them interesting on their own. Some special features of gauge theories, including the supersymmetry breaking, have been discussed in [8–10]. In three-dimensions, topologically non-trivial gauge invariant terms in the action provide masses for the gauge fields. The topological mass term introduces an infrared cutoff in vector gauge theories providing a way for the solution of the infrared problem without changing the ultraviolet behavior [10]. If absent at the classical level, this term is generated through quantum corrections [11]. In the presence of an external gauge field, fermions in three-dimensional spacetime induce a topologically nontrivial vacuum current having abnormal parity [11–17]. In models with fermions coupled to the Chern–Simons gauge field the Lorentz invariance should be spontaneously broken [18–23]: there exists a state with nonzero magnetic field and with the energy lower than the lowest energy state in the absence of

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the magnetic field. Another interesting feature of the models in two spatial dimensions is the possibility of the excitations with fractional statistics (anyons) [24].

The presence of a background gauge field gives rise to the polarization of the fermionic vacuum. As a consequence, various types of quantum numbers are generated. In particular, charge and current densities are induced [25–33]. The vacuum currents induced by cylindrical and toroidal topologies of background space have been investigated in [34, 35]. Applications are given to the electronic subsystem of graphene made cylindrical and toroidal nanotubes, described in terms of the effective Dirac-like theory. Vacuum expectation values of the current density in locally de Sitter and anti-de Sitter backgrounds with toroidally compactified spatial dimensions are studied in [36] and [37–39]. Among the most interesting topics in the studies of  $(2+1)$ -dimensional theories is the parity and chiral symmetry-breaking. In particular, it has been shown that a background magnetic field can serve as a catalyst for the dynamical symmetry breaking [40–45]. A key point of these considerations is the appearance of a nonzero fermion condensate induced by the magnetic field. This phenomenon may be important in the physics of high-temperature superconductivity [46–49].

In the present paper we investigate the finite temperature effects on the fermion condensate and on the expectation values of the charge and current densities for a massive fermionic field with nonzero chemical potential in a  $(2+1)$ -dimensional conical spacetime in the presence of a magnetic flux located at the cone apex. The corresponding vacuum expectation values in the presence of a circular boundary, concentric with the cone apex, have been studied in [50–52]. The finite temperature effects on the fermionic condensate and current densities in models with an arbitrary number of toroidally compact spatial dimensions are discussed in [53]. The finite temperature charge and current densities in the geometry of a  $(3+1)$ -dimensional cosmic string with magnetic flux have been recently investigated in [54].

The outline of the paper is as follows. In the next section we consider the fermion condensate for a two-component spinor field realizing the irreducible representation of the Clifford algebra. Expressions are derived for the separate contributions from particles and antiparticles for general values of the planar angle deficit and for the magnetic flux. Various asymptotic regions of the parameters are considered in detail. The expectation values of the charge and current densities are investigated in Sects. 3 and 4, respectively. We provide simple expressions for the topological part in the total charge, induced by the planar angle deficit and by the magnetic flux. The expectation values in the model with parity and time-reversal invariance are discussed in Sect. 5. These expectation values are obtained by combining the results from the previous section for two irreducible representations of the Clifford algebra. The main results are summarized in

Sect. 6. In the Appendix we derive the relations used in the evaluation of the topological part in the total charge.

## 2 Fermion condensate

We consider a fermionic field  $\psi(x)$  on background of  $(2+1)$ -dimensional spacetime assuming that the field is in thermal equilibrium at temperature  $T$ . In this section we evaluate the fermion condensate (FC) defined as the expectation value  $\langle \bar{\psi}\psi \rangle = \text{tr}[\hat{\rho}\bar{\psi}\psi]$ , with  $\bar{\psi} = \psi^\dagger\gamma^0$  being the Dirac adjoint and the angular brackets denote the ensemble average with the density matrix  $\hat{\rho} = Z^{-1}e^{\beta(\hat{H}-\mu'\hat{Q})}$ , where  $\beta = 1/T$ . Here  $\hat{H}$  is the Hamilton operator,  $\hat{Q}$  is a conserved charge with the related chemical potential  $\mu'$  and  $Z = \text{tr}[e^{-\beta(\hat{H}-\mu'\hat{Q})}]$ . The FC is among the most important characteristic for the system under consideration. In particular, it plays an important role in the models of dynamical breaking of chiral symmetry.

The background geometry is described by the  $(2+1)$ -dimensional line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dr^2 - r^2 d\phi^2, \quad (2.1)$$

where  $0 \leq \phi \leq \phi_0$ . For  $\phi_0 < 2\pi$ , this line element describes  $(2+1)$ -dimensional conical spacetime with the planar angle deficit  $2\pi - \phi_0$ . We will discuss the case of two-components spinor field realizing the irreducible representation of the Clifford algebra. Assuming the presence of the external electromagnetic field with the vector potential  $A_\mu$ , the field operator obeys the Dirac equation

$$(i\gamma^\mu D_\mu - sm)\psi = 0, \quad D_\mu = \partial_\mu + \Gamma_\mu + ieA_\mu, \quad (2.2)$$

where  $\Gamma_\mu$  is the spin connection and  $e$  is the charge of the field quantum. Here,  $s = +1$  and  $s = -1$  correspond to inequivalent irreducible representations of the Clifford algebra in  $(2+1)$ -dimensions (see Sect. 5). With these representations, the mass term violates the parity and time-reversal invariances. In the coordinates corresponding to (2.1), the gamma matrices can be taken in the representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^l = \frac{i^{2-l}}{r^{l-1}} \begin{pmatrix} 0 & e^{-iq\phi} \\ (-1)^{l-1}e^{iq\phi} & 0 \end{pmatrix}, \quad (2.3)$$

with  $l = 1, 2$  and  $q = 2\pi/\phi_0$ .

In what follows, in the region  $r > 0$ , we consider the gauge field configuration  $A_\mu = (0, 0, A)$ , where  $A_2 = A$  is the covariant component of the vector potential in the coordinates  $(t, r, \phi)$ . For the corresponding physical component one has  $A_\phi = -A/r$ . This corresponds to an infinitely thin magnetic flux  $\Phi = -\phi_0 A$  located at  $r = 0$ . As it will be seen below, in the expressions for the expectation values the parameter  $A$  enters in the form of the combination

$$\alpha = eA/q = -e\Phi/(2\pi),$$

where  $q = 2\pi/\phi_0$ . We decompose it as

$$\alpha = \alpha_0 + n_0, \quad |\alpha_0| < 1/2, \quad (2.4)$$

with  $n_0$  being an integer. It is the fractional part  $\alpha_0$  which is responsible for physical effects.

Let us denote by  $\psi_\sigma^{(+)}(x)$  and  $\psi_\sigma^{(-)}(x)$  a complete orthonormal set of the positive- and negative-energy solutions of the Eq. (2.2). These solutions are labeled by a set of quantum numbers  $\sigma$ . Expanding the field operator  $\psi(x)$  in terms of the functions  $\psi_\sigma^{(\pm)}(x)$  and using the commutation relations for the annihilation and creation operators, the following decomposition is obtained for the FC:

$$\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}\psi \rangle_0 + \langle \bar{\psi}\psi \rangle_+ + \langle \bar{\psi}\psi \rangle_-. \quad (2.5)$$

Here,

$$\langle \bar{\psi}\psi \rangle_0 = \sum_\sigma \bar{\psi}_\sigma^{(-)}(x) \psi_\sigma^{(-)}(x), \quad (2.6)$$

is the FC in the vacuum state and  $\langle \bar{\psi}\psi \rangle_+$  and  $\langle \bar{\psi}\psi \rangle_-$  are the contributions from particles and antiparticles, respectively. The latter are given by the expressions

$$\langle \bar{\psi}\psi \rangle_\pm = \pm \sum_\sigma \frac{\bar{\psi}_\sigma^{(\pm)}(x) \psi_\sigma^{(\pm)}(x)}{e^{\beta(E_\sigma \mp \mu)} + 1}, \quad (2.7)$$

where  $\mu = e\mu'$  and  $\pm E_\sigma$  are the energies corresponding to the modes  $\psi_\sigma^{(\pm)}(x)$ . In (2.6) and (2.7),  $\sum_\sigma$  includes the summation over the discrete quantum numbers and the integration over the continuous ones. The modes are normalized in accordance with the standard normalization condition  $\int d^2x \sqrt{\gamma} \psi_\sigma^{(\pm)\dagger}(x) \psi_\sigma^{(\pm)}(x) = \delta_{\sigma\sigma'}$ , with  $\gamma$  being the determinant of the spatial metric tensor. The part in the FC corresponding to the vacuum expectation value,  $\langle \bar{\psi}\psi \rangle_0$ , has been investigated in [51] and here we will be mainly concerned with the finite temperature effects.

For the evaluation of the FC in accordance with (2.7) we need to know the mode functions. The general solution of the radial equation for these functions contains the part regular at the origin and the part which diverges at  $r = 0$ . Under the condition

$$2|\alpha_0| \leq 1 - 1/q, \quad (2.8)$$

the irregular modes are eliminated by the normalizability condition. In the case  $2|\alpha_0| > 1 - 1/q$ , there are irregular normalizable modes and for the unique identification of the functions an additional boundary condition is required at the origin. In the literature it has been already shown that the standard procedure for the self-adjoint extension of the Dirac Hamiltonian gives rise to a one-parameter family of boundary conditions in the background of an Aharonov–Bohm gauge field [55–57]. The specific value of the parameter is related to the physical details of the magnetic field distribution inside

a more realistic finite radius flux tube (for a more detailed discussion and for specific models with finite radius magnetic flux see [58–69]). The idealized model under consideration is a limiting case of the latter.

Following [50], for irregular modes we consider a special case of boundary conditions at the cone apex, when the bag boundary condition is imposed at a finite radius, which is then taken to zero. The bag boundary condition ensures the zero flux of fermions and this corresponds to an impenetrable flux tube. The corresponding mode functions have the form

$$\psi_\sigma^{(\pm)}(x) = c_0^{(\pm)} e^{iqj\phi \mp iEt} \begin{pmatrix} J_{\beta_j}(\gamma r) e^{-iq\phi/2} \\ \epsilon_j \frac{\gamma e^{iq\phi/2}}{\pm E \pm sm} J_{\beta_j + \epsilon_j}(\gamma r) \end{pmatrix}, \quad (2.9)$$

where  $j = \pm 1/2, \pm 3/2, \dots$ ,  $J_\nu(x)$  is the Bessel function of the first kind and

$$\beta_j = q|j + \alpha| - \epsilon_j/2, \quad (2.10)$$

with  $\epsilon_j = 1$  for  $j > -\alpha$  and  $\epsilon_j = -1$  for  $j < -\alpha$ . The spinors (2.9) are specified by the set  $\sigma = (\gamma, j)$  with  $0 \leq \gamma < \infty$  and for the energy one has  $E = E_\sigma = \sqrt{\gamma^2 + m^2}$ . The normalization coefficients are given by the expression

$$c_0^{(\pm)2} = \gamma \frac{E \pm sm}{2\phi_0 E}. \quad (2.11)$$

The modes (2.9) are eigenfunctions of the angular momentum operator  $\hat{J} = -(i/q)(\partial_\phi + ieA) + \sigma_3/2$ ,  $\sigma_3 = \text{diag}(1, -1)$ , for the eigenvalues  $j + \alpha$ :

$$\hat{J} \psi_\sigma^{(\pm)}(x) = (j + \alpha) \psi_\sigma^{(\pm)}(x). \quad (2.12)$$

In the mode-sum formulas (2.6) and (2.7) one has  $\sum_\sigma = \sum_j \int_0^\infty d\gamma$ , with the summation over  $j = \pm 1/2, \pm 3/2, \dots$ . The parameter  $\alpha$  is the magnetic flux measured in units of the flux quantum. In the case when the parameter  $\alpha$  is half of an odd integer the mode with  $j = -\alpha$  should be considered separately. The corresponding mode function is presented in [50]. In order do not complicate the consideration, in the following discussion we will exclude this case. Note that, under the condition  $2|\alpha_0| > 1 - 1/q$ , the irregular mode corresponds to  $j = -n_0 - \text{sgn}(\alpha_0)/2$ . The expectation values for general case of boundary condition at the cone apex are considered in a way similar to that described below. The corresponding results differ by the contribution of the irregular mode only.

Using the mode functions (2.9), the contributions to the FC from particles and antiparticles are presented in the form

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_\pm = & \frac{q}{4\pi} \sum_j \int_0^\infty d\gamma \frac{\gamma}{e^{\beta(E \mp \mu)} + 1} \\ & \times \left\{ \frac{sm}{E} [J_{\beta_j}^2(\gamma r) + J_{\beta_j + \epsilon_j}^2(\gamma r)] \right. \\ & \left. \pm [J_{\beta_j}^2(\gamma r) - J_{\beta_j + \epsilon_j}^2(\gamma r)] \right\}. \end{aligned} \quad (2.13)$$

For the further transformation of this expression we use the integral representation of the Bessel function [70]:

$$J_\nu^2(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dt}{t} e^{t/2-x^2/t} I_\nu(x^2/t), \quad (2.14)$$

where  $c$  is a positive constant and  $I_\nu(y)$  is the modified Bessel function of the first kind. Inserting into (2.13), the separate contributions  $\langle \bar{\psi} \psi \rangle_\pm$  are expressed in terms of the series

$$\mathcal{I}(q, \alpha, z) = \sum_j I_{\beta_j}(z). \quad (2.15)$$

The function  $\mathcal{I}(q, \alpha, z)$  is periodic with respect to  $\alpha$  with the period 1. Hence, if we present the parameter  $\alpha$  in the form (2.4), then  $\mathcal{I}(q, \alpha, z)$  does not depend on  $n_0$  and  $\mathcal{I}(q, \alpha, z) = \mathcal{I}(q, \alpha_0, z)$ . For the second series, appearing after the application of (2.14), one has  $\sum_j I_{\beta_j+\epsilon_j}(z) = \mathcal{I}(q, -\alpha_0, z)$ . As a result, the expression on the right of (2.13) is transformed as

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_\pm &= \frac{q}{4\pi} \int_0^\infty d\gamma \frac{\gamma}{e^{\beta(E \mp \mu)} + 1} \\ &\times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dt}{t} e^{t/2-\gamma^2 r^2/t} \\ &\times \sum_{\delta=\pm 1} (sm/E \pm \delta) \mathcal{I}(q, \delta\alpha_0, \gamma^2 r^2/t). \end{aligned} \quad (2.16)$$

For the series (2.15) one has the representation [50]

$$\begin{aligned} \mathcal{I}(q, \alpha_0, x) &= \frac{2}{q} \sum_{l=0}^p (-1)^l \cos(\pi l(2\alpha_0 - 1/q)) e^{x \cos(2\pi l/q)} \\ &- \frac{1}{\pi} \int_0^\infty dy \frac{e^{-x \cosh y} f(q, \alpha_0, y)}{\cosh(qy) - \cos(q\pi)}, \end{aligned} \quad (2.17)$$

where  $p$  is the integer part of  $q/2$ ,  $p = [q/2]$ , and

$$\begin{aligned} f(q, \alpha_0, y) &= \sum_{\delta=\pm 1} \delta \cos(q\pi(1/2 - \delta\alpha_0)) \\ &\times \cosh((q\alpha_0 + \delta q/2 - 1/2)y). \end{aligned} \quad (2.18)$$

The prime on the sign of the summation in (2.17) means that the term  $l = 0$  and the term  $l = q/2 = p$  for even values of  $q$  should be taken with the coefficient  $1/2$ . In the range  $1 \leq q < 2$ , the sum over  $l$  in (2.17) is absent. Note that for  $q$  being an even integer,  $q = 2p$ , for the function (2.18) one has

$$\begin{aligned} f(q, \alpha_0, y) &= 2(-1)^p \cos(\pi q\alpha_0) \sinh \\ &((q\alpha_0 - 1/2)y) \sinh(py), \end{aligned} \quad (2.19)$$

and the integrand in (2.17) is regular in the lower limit of integration.

Substituting the representation (2.17) into (2.16), the  $t$ -integral is evaluated with the help of the formula [70]

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dt}{t} e^{t/2-2b^2/t} = J_0(2b). \quad (2.20)$$

As a result, the following expression is obtained

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_\pm &= \langle \bar{\psi} \psi \rangle_\pm^{(M)} + \frac{1}{\pi} \int_0^\infty d\gamma \frac{\gamma}{e^{\beta(E \mp \mu)} + 1} \\ &\times \left\{ \sum_{l=1}^p (-1)^l \left[ \frac{sm}{E} c_l \cos(2\pi l\alpha_0) \right. \right. \\ &\left. \left. \pm s_l \sin(2\pi l\alpha_0) \right] J_0(2\gamma r s_l) \right. \\ &\left. - \frac{q}{\pi} \int_0^\infty dy \frac{\frac{sm}{E} f_1(q, \alpha_0, y) \pm f_2(q, \alpha_0, y)}{\cosh(2qy) - \cos(q\pi)} \right. \\ &\left. \times J_0(2\gamma r \cosh y) \right\}, \end{aligned} \quad (2.21)$$

with the notations

$$c_l = \cos(\pi l/q), \quad s_l = \sin(\pi l/q), \quad (2.22)$$

and

$$\begin{aligned} f_1(q, \alpha_0, y) &= -\sinh y \sum_{\delta=\pm 1} \cos(q\pi(1/2 - \delta\alpha_0)) \\ &\times \sinh((1 + 2\delta\alpha_0)qy), \\ f_2(q, \alpha_0, y) &= \cosh y \sum_{\delta=\pm 1} \delta \cos(q\pi(1/2 - \delta\alpha_0)) \\ &\times \cosh((1 + 2\delta\alpha_0)qy). \end{aligned} \quad (2.23)$$

Note that we have  $f_l(q, \alpha_0, y) = \sum_{n=\pm 1} n^{l-1} f(q, n\alpha_0, 2y)/2$ ,  $n = 1, 2$ . In (2.21), the part

$$\langle \bar{\psi} \psi \rangle_\pm^{(M)} = \frac{sm}{2\pi} \int_0^\infty d\gamma \frac{\gamma/E}{e^{\beta(E \mp \mu)} + 1}, \quad (2.24)$$

is the contribution coming from the  $l = 0$  term in the right-hand side of (2.17) and it coincides with the corresponding quantity in (2 + 1)-dimensional Minkowski spacetime ( $q = 1$ ) in the absence of the magnetic flux ( $\alpha_0 = 0$ ).

For an integer  $q$ , the expression (2.21) is essentially simplified for special values of  $\alpha_0$  defined by

$$\alpha_0 = \frac{n + 1/2 - \{q/2\}}{q}, \quad (2.25)$$

where  $n$  is an integer and the figure braces stand for the fractional part of the enclosed expression. The allowed values for  $n$  are obtained from the condition  $|\alpha_0| < 1/2$ . For  $\alpha_0$  from (2.25) one has  $\cos(q\pi(1/2 - \delta\alpha_0)) = 0$  and, hence, both the functions  $f_l(q, \alpha_0, y)$  in (2.23) become zero. As a result, the integral term in the figure braces of (2.21) vanishes.

Returning to the general values of the parameters  $q$  and  $\alpha_0$ , first we will consider the case with  $|\mu| \leq m$ . By using the relation

$$(e^y + 1)^{-1} = - \sum_{n=1}^{\infty} (-1)^n e^{-ny}, \quad (2.26)$$

after the evaluation of the  $\gamma$ -integrals, the following representation is obtained

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_{\pm} &= \langle \bar{\psi} \psi \rangle_{\pm}^{(M)} - \frac{4m^2}{(2\pi)^{3/2}} \sum_{n=1}^{\infty} (-1)^n e^{\pm n\beta\mu} \\ &\times \left\{ \sum_{l=1}^p (-1)^l \left[ s c_l \cos(2\pi l \alpha_0) f_{1/2}(c_{nl}) \right. \right. \\ &\pm m n \beta s_l \sin(2\pi l \alpha_0) f_{3/2}(c_{nl}) \left. \right] \\ &- \frac{q}{\pi} \int_0^{\infty} dy \left[ \frac{s f_1(q, \alpha_0, y) f_{1/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right. \\ &\left. \left. \pm m n \beta \frac{f_2(q, \alpha_0, y) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right] \right\}, \quad (2.27) \end{aligned}$$

where  $f_v(x) = K_v(x)/x^v$ , with  $K_v(x)$  being the Macdonald function, and

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_{\pm}^{(M)} &= -\frac{sm}{2\pi\beta} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{(\pm\mu-m)n\beta} \\ &= \frac{sm}{2\pi\beta} \ln \left( 1 + e^{(\pm\mu-m)\beta} \right). \quad (2.28) \end{aligned}$$

In (2.27), we have introduced the notations

$$\begin{aligned} c_{nl} &= m \sqrt{n^2 \beta^2 + 4r^2 \sin^2(\pi l/q)}, \\ c_n(y) &= m \sqrt{n^2 \beta^2 + 4r^2 \cosh^2 y}. \quad (2.29) \end{aligned}$$

For the functions  $f_v(x)$  in (2.27) we have

$$f_{1/2}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x}, \quad f_{3/2}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{x^3} (1+x). \quad (2.30)$$

Now summing all the contributions, for the FC we get

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= \langle \bar{\psi} \psi \rangle_0 + \langle \bar{\psi} \psi \rangle^{(M)} - \frac{2^{3/2} m^2}{\pi^{3/2}} \sum_{n=1}^{\infty} (-1)^n \\ &\times \left\{ \cosh(n\beta\mu) \left[ \sum_{l=1}^p (-1)^l \right. \right. \\ &\times s c_l \cos(2\pi l \alpha_0) f_{1/2}(c_{nl}) - \frac{s q}{\pi} \\ &\times \int_0^{\infty} dy \frac{f_1(q, \alpha_0, y) f_{1/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \left. \right] \\ &+ m n \beta \sinh(n\beta\mu) \left[ \sum_{l=1}^p (-1)^l s_l \sin(2\pi l \alpha_0) f_{3/2}(c_{nl}) \right. \\ &\left. \left. - \frac{q}{\pi} \int_0^{\infty} dy \frac{f_2(q, \alpha_0, y) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right] \right\}, \quad (2.31) \end{aligned}$$

with the Minkowskian part

$$\langle \bar{\psi} \psi \rangle^{(M)} = \frac{sm}{2\pi\beta} \left[ \ln(1 + e^{-\beta(m-\mu)}) + \ln(1 + e^{-\beta(m+\mu)}) \right]. \quad (2.32)$$

The expression for the FC in the vacuum state can be found in [51]:

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_0 &= -\frac{sm}{2\pi r} \left[ \sum_{l=1}^p (-1)^l \frac{c_l}{s_l} \cos(2\pi l \alpha_0) e^{-2mr s_l} \right. \\ &\left. - \frac{q}{\pi} \int_0^{\infty} dy \frac{f_1(q, \alpha_0, y)}{\cosh(2qy) - \cos(q\pi)} \frac{e^{-2mr \cosh y}}{\cosh y} \right]. \quad (2.33) \end{aligned}$$

This contribution can be combined with the sum over  $n$  in (2.31) containing the factor  $\cosh(n\beta\mu)$  writing  $\sum_{n=0}^{\infty'}$  instead of  $\sum_{n=1}^{\infty}$ . The prime on the summation sign means that the term  $n = 0$  should be taken with the coefficient  $1/2$ . The FC is a periodic function of the magnetic flux with the period equal to the flux quantum. Note that it has no definite parity with respect to the reflections  $\alpha_0 \rightarrow -\alpha_0$  and  $\mu \rightarrow -\mu$ . This is related to the non-invariance of the model under the parity ( $P$ -) and time-reversal ( $T$ -) transformations (see the discussion in Sect. 5). Note that the FC in the vacuum state is an even function of  $\alpha_0$ .

The general formula (2.31) is further simplified in various special cases. For a massless fermionic field, because of the condition  $|\mu| \leq m$ , we should also assume that  $\mu = 0$  and the FC vanishes. In the special case when the magnetic flux is absent, one has  $\alpha_0 = 0$ , and the Eq. (2.31) becomes

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= \langle \bar{\psi} \psi \rangle^{(M)} - \frac{2sm^2}{\pi} \sum_{n=0}^{\infty'} (-1)^n \cosh(n\beta\mu) \\ &\times \left[ \sum_{l=1}^p (-1)^l c_l \frac{e^{-c_{nl}}}{c_{nl}} + \frac{2q}{\pi} \cos(q\pi/2) \right. \\ &\times \left. \int_0^{\infty} dy \frac{\sinh(2qy) \sinh y}{\cosh(2qy) - \cos(q\pi)} \frac{e^{-c_n(y)}}{c_n(y)} \right]. \quad (2.34) \end{aligned}$$

In this case the FC is an even function of the chemical potential. For the background of Minkowski spacetime with the magnetic flux we take  $q = 1$  and the FC is given by

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= \langle \bar{\psi} \psi \rangle^{(M)} + \frac{2^{3/2} m^2}{\pi^{5/2}} \sin(\pi \alpha_0) \sum_{n=0}^{\infty'} (-1)^n \int_0^{\infty} dy \\ &\times \left[ -s \cosh(n\beta\mu) \tanh y \sinh(2\alpha_0 y) f_{1/2}(c_n(y)) \right. \\ &\left. + m n \beta \sinh(n\beta\mu) \cosh(2\alpha_0 y) f_{3/2}(c_n(y)) \right]. \quad (2.35) \end{aligned}$$

The  $n = 0$  terms in (2.34) and (2.35) correspond to the vacuum expectation values.

Let us consider the asymptotic behavior of the FC in the limiting regions of the parameters. For points near the apex of the cone two separate regions of the values of the parameter  $\alpha_0$  should be distinguished. For  $2|\alpha_0| < 1 - 1/q$ , the thermal part is finite on the apex and the corresponding expression



is directly obtained from (2.31) putting  $r = 0$ . Recall that, in this case, all the modes (2.9) are regular at the apex. For  $2|\alpha_0| > 1 - 1/q$  the integrals diverge on the apex. In order to find the leading term in the asymptotic expansion over the distance from the origin, we note that for points near the apex the dominant contribution to the integrals in (2.31) comes from large values of  $y$ . Expanding the integrands, it can be seen that

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &\approx \langle \bar{\psi} \psi \rangle_0 + \frac{m^2 q \Gamma(1/2 - \rho)}{2^\rho \pi^{5/2} (mr)^{1-2\rho}} \cos(\pi\rho) \sum_{n=1}^{\infty} (-1)^n \\ &\times [\operatorname{sgn}(\alpha_0) mn\beta \sinh(n\beta\mu) f_{\rho+1}(mn\beta) \\ &- s \cosh(n\beta\mu) f_{\rho}(mn\beta)], \end{aligned} \quad (2.36)$$

with the notation  $\rho = q(1/2 - |\alpha_0|)$ . Note that in the case under consideration one has  $\rho < 1/2$ . Hence, for  $2|\alpha_0| > 1 - 1/q$  the finite temperature part diverges on the apex as  $r^{2\rho-1}$ . The vacuum expectation value  $\langle \bar{\psi} \psi \rangle_0$  behaves as  $1/r$  and it dominates for small  $r$ .

Now we turn to the investigation of the asymptotics for the FC at low and high temperatures. In the low temperature limit the main contribution to the thermal part comes from the  $n = 1$  term and to the leading order we find

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &\approx \langle \bar{\psi} \psi \rangle_0 + \frac{m}{\pi\beta} e^{-\beta(m-|\mu|)} \\ &\times \left[ \sum_{l=0}^p (-1)^l \cos(\pi l(2s\alpha_0 \operatorname{sgn}(\mu) - 1/q)) \right. \\ &- \frac{q^s}{\pi} \operatorname{sgn}(\mu) \sum_{\delta=\pm 1} \delta \cos(q\pi(1/2 - \delta\alpha_0)) \\ &\left. \int_0^\infty dy \frac{\cosh((q + 2\delta\alpha_0 q - s\delta \operatorname{sgn}(\mu))y)}{\cosh(2qy) - \cos(q\pi)} \right]. \end{aligned} \quad (2.37)$$

Here, the term with  $l = 0$  is the contribution of the Minkowskian part  $\langle \bar{\psi} \psi \rangle^{(M)}$ . Hence, for  $|\mu| < m$ , at low temperatures the thermal contribution in the FC is suppressed by the factor  $e^{-\beta(m-|\mu|)}$ .

The representation (2.31) is not well adapted for the investigation of the high temperature limit. An alternative representation, is obtained by using the relations  $(-1)^n \cosh(n\beta\mu) = \cos(nb)$  and  $(-1)^n n \sinh(n\beta\mu) = \partial_\mu \cos(nb)/\beta$ , with  $b = \pi + i\beta\mu$ , and then the formula [34,35]

$$\begin{aligned} &\sum_{n=0}^{\infty} \cos(nb) f_v(m\sqrt{\beta^2 n^2 + a^2}) \\ &= \frac{(2\pi)^{1/2}}{2\beta m^{2v}} \sum_{n=-\infty}^{+\infty} \left[ (2\pi n + b)^2 \beta^{-2} + m^2 \right]^{v-1/2} \\ &\times f_{v-1/2} \left( a \sqrt{(2\pi n + b)^2 \beta^{-2} + m^2} \right), \end{aligned} \quad (2.38)$$

for the series over  $n$  in (2.31). With these transformations, the FC is presented as

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= \langle \bar{\psi} \psi \rangle^{(M)} - \frac{2mT}{\pi} \sum_{n=-\infty}^{\infty} \\ &\times \left\{ \left[ \sum_{l=1}^p (-1)^l s c_l \cos(2\pi l \alpha_0) \right. \right. \\ &\times K_0(2r s_l b_n) - \frac{q}{\pi} \int_0^\infty dy \\ &\times \frac{s f_1(q, \alpha_0, y) K_0(2r b_n \cosh y)}{\cosh(2qy) - \cos(q\pi)} \left. \right] \\ &+ \frac{\mu - i\pi(2n+1)T}{m} \left[ \sum_{l=1}^p (-1)^l s_l \sin(2\pi l \alpha_0) K_0(2r s_l b_n) \right. \\ &\left. \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) K_0(2r b_n \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right] \right\}, \end{aligned} \quad (2.39)$$

where we have introduced the notation

$$b_n = \{[\pi(2n+1)T + i\mu]^2 + m^2\}^{1/2}. \quad (2.40)$$

In the case of zero chemical potential,  $\mu = 0$ , the expression (2.39) is simplified to

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= \frac{smT}{\pi} \left\{ \ln(1 + e^{-m\beta}) - 4 \sum_{n=0}^{\infty} \right. \\ &\times \left[ \sum_{l=1}^p (-1)^l c_l \cos(2\pi l \alpha_0) \right. \\ &\times K_0(2r s_l b_n^{(0)}) - \frac{q}{\pi} \int_0^\infty dy \\ &\times \frac{f_1(q, \alpha_0, y) K_0(2r b_n^{(0)} \cosh y)}{\cosh(2qy) - \cos(q\pi)} \left. \right] \left. \right\}, \end{aligned} \quad (2.41)$$

with  $b_n^{(0)} = \sqrt{[\pi(2n+1)T]^2 + m^2}$ .

In the high temperature limit,  $Tr \gg 1$ , the dominant contribution to the series over  $n$  in (2.39) comes from the terms with  $n = 0$  and  $n = -1$  and from the part containing the factor  $\mu - i\pi(2n+1)T$ . In the case  $q > 2$  the leading contribution corresponds to the term with  $l = 1$  and we obtain

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &\approx \langle \bar{\psi} \psi \rangle^{(M)} - 2\sqrt{s_1/r} \sin(2\pi\alpha_0) \sin(2r s_1 \mu) T^{3/2} \\ &\times e^{-2\pi s_1 Tr}. \end{aligned} \quad (2.42)$$

For  $q < 2$  the sums over  $l$  are absent. At high temperatures the dominant contribution to the integrals in (2.39) comes from the region near the lower limit of integration. Assuming that  $T \gg 1/[\pi r \sin^2(\pi q/2)]$  ( $q$  is not too close to 2), to the leading order one finds

$$\langle \bar{\psi} \psi \rangle \approx \langle \bar{\psi} \psi \rangle^{(M)} - \frac{q \sin(q\pi\alpha_0)}{\pi r \sin(\pi q/2)} \sin(2\mu r) T e^{-2\pi Tr}, \quad (2.43)$$

and the suppression is stronger. The high-temperature asymptotic for the Minkowskian part is given by

$$\langle \bar{\psi} \psi \rangle^{(M)} \approx \frac{smT}{2\pi} \ln 2, \quad (2.44)$$

and, at high temperatures, the FC is dominated by this part. The contributions induced by the conical defect and the magnetic flux are exponentially small for points not close to the origin.

For the investigation of the FC at large distances from the origin it is convenient to use the representation (2.39). The corresponding procedure is similar to that for the high temperature asymptotic and the dominant contribution comes from the terms with  $n = 0$  and  $n = -1$ . Considering the asymptotic expression for the Macdonald function with large arguments, for  $q > 2$  we obtain

$$\langle \bar{\psi} \psi \rangle \approx \langle \bar{\psi} \psi \rangle^{(M)} + 2\sqrt{\pi s_1/r} \sin(2\pi\alpha_0) T^2 \times \text{Im} \left( b_0^{-1/2} e^{-2rs_1 b_0} \right). \quad (2.45)$$

where  $b_0 = \sqrt{(\pi T + i\mu)^2 + m^2}$ . In the case  $q < 2$ , assuming that  $\sin(\pi q/2) \gg q/r\sqrt{T^2 + m^2}$ , the asymptotic expression is given by

$$\langle \bar{\psi} \psi \rangle \approx \langle \bar{\psi} \psi \rangle^{(M)} + \frac{qT \sin(q\pi\alpha_0)}{\pi r \sin(\pi q/2)} \times \text{Re} \left[ (\mu - i\pi T) e^{-2rb_0/b_0} \right]. \quad (2.46)$$

In both the cases, the parts in the FC induced by the conical defect and by the magnetic flux are exponentially small and in the leading order we have  $\langle \bar{\psi} \psi \rangle \approx \langle \bar{\psi} \psi \rangle^{(M)}$ .

Now let us turn to the case  $|\mu| > m$ . For  $\mu > m$  ( $\mu < -m$ ) the contribution of the antiparticles (particles) to the FC is evaluated in a way similar to that described above and the corresponding expression is given by (2.27) with the lower (upper) sign. The total FC is expressed as

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= \langle \bar{\psi} \psi \rangle_0 + \langle \bar{\psi} \psi \rangle_{\pm} + \frac{1}{\pi} \int_0^\infty d\gamma \frac{\gamma}{e^{\beta(E-|\mu|)} + 1} \\ &\times \left\{ \sum_{l=0}^p (-1)^l \left[ \frac{sm}{E} c_l \cos(2\pi l\alpha_0) \mp s_l \sin(2\pi l\alpha_0) \right] J_0(2\gamma r s_l) \right. \\ &\left. - \frac{q}{\pi} \int_0^\infty dy \frac{\frac{sm}{E} f_1(q, \alpha_0, y) \mp f_2(q, \alpha_0, y)}{\cosh(2qy) - \cos(q\pi)} J_0(2\gamma r \cosh y) \right\}. \end{aligned} \quad (2.47)$$

where the upper and lower signs correspond to  $\mu < -m$  and  $\mu > m$ , respectively. Introducing the Fermi momentum  $p_0 = \sqrt{\mu^2 - m^2}$ , the integration over  $\gamma$  in the last term can be divided into two regions with  $\gamma < p_0$  and  $\gamma > p_0$ . The expansion (2.26) can be further applied to the integral over the second region. At high temperatures,  $T \gg |\mu|$ , the dominant contribution to the FC comes from the states with the energies  $E \gg |\mu|$  and the asymptotic estimates considered above for the case  $|\mu| < m$  are still valid.

Compared to the case  $|\mu| < m$ , the situation in the range  $|\mu| > m$  is completely different at low temperatures. In the limit  $T \rightarrow 0$ , in the part coming from the particles or antiparticles, the contribution of the states with  $E \leq |\mu|$  survives only and we find

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_{T=0} &= \langle \bar{\psi} \psi \rangle_0 + \frac{p_0^2}{\pi} \left\{ \sum_{l=0}^p (-1)^l \right. \\ &\times [s c_l \cos(2\pi l\alpha_0) g_1(p_0 r s_l) + \text{sgn}(\mu) s_l \sin(2\pi l\alpha_0) g_2(p_0 r s_l)] \\ &\left. - \frac{q}{\pi} \int_0^\infty dy \frac{\sum_{l=1}^2 s^l (\text{sgn}(\mu))^{l-1} f_l(q, \alpha_0, y) g_l(p_0 r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right\}. \end{aligned} \quad (2.48)$$

with the notations

$$g_1(u) = \frac{m}{p_0} \int_0^1 dx \frac{x J_0(2ux)}{\sqrt{x^2 + m^2/p_0^2}}, \quad g_2(u) = \frac{J_1(2u)}{2u}. \quad (2.49)$$

The second term in the right-hand side of (2.48) is the contribution of particles (for  $\mu > m$ ) or antiparticles (for  $\mu < -m$ ) filling the states with the energies  $m \leq E \leq |\mu|$ . This term is finite on the apex for  $2|\alpha_0| < 1 - 1/q$  and diverges as  $1/r^{1-2\rho}$  for  $2|\alpha_0| > 1 - 1/q$ . The vacuum contribution near the apex behaves as  $1/r$  and it dominates. At large distances from the apex,  $p_0 r \gg 1$ , the corrections induced by the conical defect and by the magnetic flux are dominated by the part coming from the particles or antiparticles (the second term in the right-hand side of (2.48) with the contribution of  $l = 0$  excluded). This part decays oscillatory with the amplitude decreasing as  $1/r^{3/2}$ . Note that the decay of the vacuum part for a massive field is exponential. In the case of a massless field,  $m = 0$ , the vacuum part of the FC vanishes and the formula (2.48) simplifies to

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_{T=0} &= \text{sgn}(\mu) \frac{\mu^2}{\pi} \left[ \sum_{l=0}^p (-1)^l s_l \sin(2\pi l\alpha_0) g_2(|\mu| r s_l) \right. \\ &\left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) g_2(|\mu| r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right]. \end{aligned} \quad (2.50)$$

In this case, the only nonzero contribution comes from particles or antiparticles and the FC is an odd function of both the chemical potential and the magnetic flux.

In the discussion above we have assumed that the field  $\psi(x)$  is periodic along the azimuthal direction. We may consider a more general case where the spinor field obeys the quasiperiodicity condition

$$\psi(t, r, \phi + \phi_0) = e^{i\chi} \psi(t, r, \phi), \quad (2.51)$$

with a constant phase  $\chi$ . The corresponding expressions for the expectation values are obtained from those presented above (and in what follows) with  $\alpha$  given by the formula

$$\alpha = \chi/(2\pi) + eA/q. \quad (2.52)$$

Note that  $\chi$  and  $A$  are changed by a gauge transformation whereas their combination in the right-hand side of (2.52) is gauge invariant.

### 3 Charge density

In this and in the following section we shall investigate the expectation value of the fermionic current density given by  $\langle j^\nu \rangle = e \operatorname{tr} [\hat{\rho} \tilde{\psi}(x) \gamma^\nu \psi(x)]$ . Similarly to the case of the FC, the current density is decomposed as

$$\langle j^\nu \rangle = \langle j^\nu \rangle_0 + \langle j^\nu \rangle_+ + \langle j^\nu \rangle_-, \quad (3.1)$$

where

$$\langle j^\nu \rangle_0 = e \sum_\sigma \tilde{\psi}_\sigma^{(-)}(x) \gamma^\nu \psi_\sigma^{(-)}(x), \quad (3.2)$$

is the vacuum expectation value and

$$\langle j^\nu \rangle_\pm = \pm e \sum_\sigma \frac{\tilde{\psi}_\sigma^{(\pm)} \gamma^\nu \psi_\sigma^{(\pm)}}{e^{\beta(E_\sigma \mp \mu)} + 1}. \quad (3.3)$$

The terms  $\langle j^\nu \rangle_+$  and  $\langle j^\nu \rangle_-$  are the contributions to the current density coming from particles and antiparticles. The vacuum expectation value has been investigated in [50] and here we will be focused with the finite temperature effects. The details of the calculations are similar to those for the FC and the main steps only will be given.

We start with the charge density that corresponds to the component  $\nu = 0$  in (3.1). By using the mode functions (2.9), for the contributions from the particles and antiparticles we get the representation

$$\begin{aligned} \langle j^0 \rangle_\pm = & \pm \frac{eq}{4\pi} \sum_j \int_0^\infty d\gamma \frac{\gamma}{e^{\beta(E \mp \mu)} + 1} \\ & \times \left\{ J_{\beta_j}^2(\gamma r) + J_{\beta_j + \epsilon_{\beta_j}}^2(\gamma r) \pm \frac{sm}{E} \right. \\ & \times \left. \left[ J_{\beta_j}^2(\gamma r) - J_{\beta_j + \epsilon_{\beta_j}}^2(\gamma r) \right] \right\}. \end{aligned} \quad (3.4)$$

As we could expect  $\pm \langle j^0 \rangle_\pm / e > 0$ . By the transformations similar to those for the FC, one finds the following expression

$$\begin{aligned} \langle j^0 \rangle_\pm = & \langle j^0 \rangle_\pm^{(M)} + \frac{e}{\pi} \int_0^\infty d\gamma \frac{\gamma}{e^{\beta(E \mp \mu)} + 1} \\ & \times \left\{ \sum_{l=1}^p (-1)^l \left[ \frac{sm}{E} s_l \sin(2\pi l \alpha_0) \right. \right. \\ & \left. \left. \pm c_l \cos(2\pi l \alpha_0) \right] J_0(2\gamma r s_l) \right. \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \frac{\frac{sm}{E} f_2(q, \alpha_0, 2y) \pm f_1(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} \right. \\ & \left. \times J_0(2\gamma r \cosh y) \right\}, \end{aligned} \quad (3.5)$$

with the Minkowskian part

$$\langle j^0 \rangle_\pm^{(M)} = \pm \frac{e}{2\pi} \int_0^\infty d\gamma \frac{\gamma}{e^{\beta(E \mp \mu)} + 1}. \quad (3.6)$$

For a massless field and in the case of the zero chemical potential,  $\mu = 0$ , the contributions from the particles and antiparticles to the total charge density cancel each other:  $\langle j^0 \rangle_+ = -\langle j^0 \rangle_-$ . This is not the case for a massive field.

In the case  $|\mu| \leq m$ , by using the relation (2.26), we get

$$\begin{aligned} \langle j^0 \rangle_\pm = & \langle j^0 \rangle_\pm^{(M)} - \frac{\sqrt{2}em^2}{\pi^{3/2}} \sum_{n=1}^\infty (-1)^n e^{\pm n\beta\mu} \\ & \times \left\{ \sum_{l=1}^p (-1)^l \left[ s s_l \sin(2\pi l \alpha_0) f_{1/2}(c_{nl}) \right. \right. \\ & \left. \left. \pm mn\beta c_l \cos(2\pi l \alpha_0) f_{3/2}(c_{nl}) \right] \right. \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \left[ \frac{s f_2(q, \alpha_0, y) f_{1/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right. \right. \\ & \left. \left. \pm \frac{mn\beta f_1(q, \alpha_0, y) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right] \right\}, \end{aligned} \quad (3.7)$$

with

$$\langle j^0 \rangle_\pm^{(M)} = \mp \frac{em^3\beta}{\sqrt{2}\pi^{3/2}} \sum_{n=1}^\infty (-1)^n e^{\pm n\beta\mu} f_{3/2}(mn\beta). \quad (3.8)$$

The vacuum expectation value of the charge density is given by the expression [50]

$$\begin{aligned} \langle j^0 \rangle_0 = & -\frac{sem}{2\pi r} \left[ \sum_{l=1}^p (-1)^l \sin(2\pi l \alpha_0) e^{-2mr s_l} \right. \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y)}{\cosh(2qy) - \cos(q\pi)} \frac{e^{-2mr \cosh y}}{\cosh y} \right]. \end{aligned} \quad (3.9)$$

For the case of a massless field, because of the condition  $|\mu| \leq m$ , in (3.7) we should also assume  $\mu = 0$ . By taking



into account that  $f_\nu(x) \approx 2^{\nu-1} \Gamma(\nu) x^{-2\nu}$  for  $x \rightarrow 0$ , we find the expression

$$\langle j^0 \rangle_\pm = \pm \frac{\pi e T^2}{24} \mp \frac{e}{\pi T} \sum_{n=1}^{\infty} (-1)^n n \left[ \sum_{l=1}^p \frac{(-1)^l c_l \cos(2\pi l \alpha_0)}{(n^2 \beta^2 + 4r^2 s_l^2)^{3/2}} - \frac{q}{\pi} \int_0^\infty dy \frac{f_1(q, \alpha_0, y) (n^2 \beta^2 + 4r^2 \cosh^2 y)^{-3/2}}{\cosh(2qy) - \cos(q\pi)} \right], \quad (3.10)$$

where the first term in the right-hand side is the Minkowskian part. In this case the vacuum charge density vanishes. Note that for a massless field the charge densities (3.10) do not depend on the parameter  $s$ . They are even functions of  $\alpha_0$ .

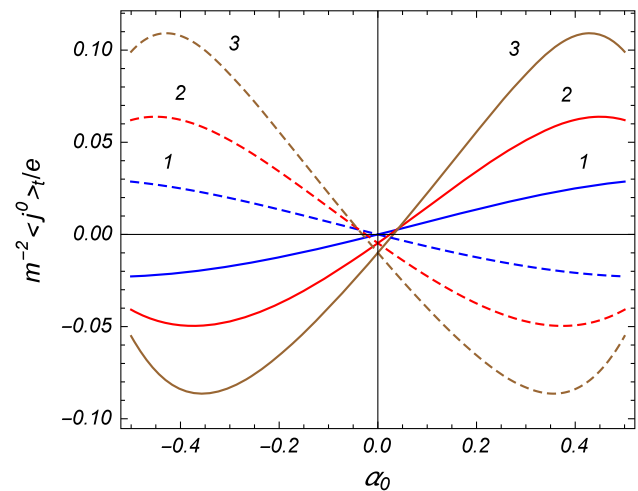
Summing the contributions from the vacuum expectation value and from particles and antiparticles, for the total charge density, in the case  $|\mu| \leq m$ , we obtain

$$\begin{aligned} \langle j^0 \rangle &= \langle j^0 \rangle^{(M)} - \frac{2^{3/2} e m^2}{\pi^{3/2}} \left\{ \sum_{n=0}^{\infty} (-1)^n s \cosh(n\beta\mu) \right. \\ &\quad \times \left[ \sum_{l=1}^p (-1)^l s_l \sin(2\pi l \alpha_0) f_{1/2}(c_{nl}) \right. \\ &\quad \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) f_{1/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right] \\ &\quad + m\beta \sum_{n=1}^{\infty} (-1)^n n \sinh(n\beta\mu) \\ &\quad \times \left[ \sum_{l=1}^p (-1)^l c_l \cos(2\pi l \alpha_0) f_{3/2}(c_{nl}) \right. \\ &\quad \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_1(q, \alpha_0, y) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right] \Big\}, \quad (3.11) \end{aligned}$$

with the Minkowskian term

$$\langle j^0 \rangle^{(M)} = -\frac{\sqrt{2} e m^3}{\pi^{3/2} T} \sum_{n=1}^{\infty} (-1)^n n \sinh(n\beta\mu) f_{3/2}(mn\beta). \quad (3.12)$$

Note that, in the case of zero chemical potential, the Minkowskian part in the charge density vanishes as a consequence of the cancellation of the contributions coming from particles and antiparticles. The magnetic flux acts on these contributions in different ways and there is no such a cancellation for the topological part. The  $n = 0$  term in (3.11) corresponds to the vacuum expectation value of the charge density. In the case of a massless field with the zero chemical potential the charge density vanishes. Note that, for a massive field, the charge density has indefinite parity with respect to the change of the sign for  $\alpha_0$ ,  $\alpha_0 \rightarrow -\alpha_0$ , and for  $\mu$ ,  $\mu \rightarrow -\mu$ . This is related to the fact that, in the presence of



**Fig. 1** Topological part in the charge density as a function of the parameter  $\alpha_0$  for fields realizing the irreducible representations with  $s = 1$  (full curves) and  $s = -1$  (dashed curves). The numbers near the curves are the corresponding values of  $q$ . The graphs are plotted for  $\mu/m = 0.25$ ,  $mr = 0.5$ , and  $T/m = 0.5$

the mass term, the model under consideration is not invariant under the  $T$ - and  $P$ -transformations.

The quantity  $\langle j^0 \rangle_t = \langle j^0 \rangle - \langle j^0 \rangle^{(M)}$  gives the contribution to the charge density coming from the planar angle deficit and from the magnetic flux. For brevity we shall call it the topological part. In Fig. 1 we have plotted this part versus the parameter  $\alpha_0$ . Recall that the charge density is a periodic function of  $\alpha_0$  with the period 1. The full and dashed curves correspond to the irreducible representation with  $s = 1$  and  $s = -1$ , respectively. The numbers near the curves are the values of  $q$ . The graphs are plotted for  $\mu/m = 0.25$ ,  $mr = 0.5$ ,  $T/m = 0.5$ . For this example, the contribution of the terms in (3.11) containing the factor  $s$  dominates. These terms are odd with respect to the reflection  $\alpha_0 \rightarrow -\alpha_0$ . Note that for the same values of  $\mu/m = 0.25$ ,  $T/m = 0.5$ , for the Minkowskian part one has  $\langle j^0 \rangle^{(M)} \approx 0.015 e m^2$ .

The general formula (3.11) is simplified in special cases. For integer values of  $q$  and for the values of  $\alpha_0$  given by (2.25), the integral terms vanish. In the absence of the magnetic flux and for general values of  $q$ , the charge density is given by

$$\begin{aligned} \langle j^0 \rangle &= -\frac{e m^3 \beta}{(\pi/2)^{3/2}} \sum_{n=1}^{\infty} (-1)^n n \sinh(n\beta\mu) \left[ \sum_{l=0}^p (-1)^l c_l f_{3/2}(c_{nl}) \right. \\ &\quad \left. + \frac{2q}{\pi} \cos(q\pi/2) \int_0^\infty dy \frac{\sinh y \sinh(qy) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right], \quad (3.13) \end{aligned}$$

and it is an odd function of the chemical potential. For the charge density in the background of Minkowski spacetime ( $q = 1$ ) in the presence of a magnetic flux we get

$$\begin{aligned} \langle j^0 \rangle = & \langle j^0 \rangle^{(M)} + \frac{\sqrt{2}em^2}{\pi^{5/2}} \sin(\pi\alpha_0) \sum_{n=0}^{\infty} (-1)^n \\ & \times \left[ s \cosh(n\beta\mu) \int_0^\infty dy \cosh(2\alpha_0 y) f_{1/2}(c_n(y)) \right. \\ & \left. - nm\beta \sinh(n\beta\mu) \int_0^\infty dy \tanh y \sinh(2\alpha_0 y) f_{3/2}(c_n(y)) \right]. \end{aligned} \quad (3.14)$$

For  $2|\alpha_0| < 1 - 1/q$ , the finite temperature part in the charge density is finite at the cone apex. This part is given by the right-hand side of (3.11), excluding the term  $n = 0$ . The corresponding expression is obtained by the direct substitution  $r = 0$ . In the case  $2|\alpha_0| > 1 - 1/q$ , the analysis similar to that for the FC leads to the following asymptotic expression

$$\begin{aligned} \langle j^0 \rangle \approx & \langle j^0 \rangle_0 + \frac{em^2 q \Gamma(1/2 - \rho)}{2\rho \pi^{5/2} (mr)^{1-2\rho}} \cos(\pi\rho) \sum_{n=1}^{\infty} (-1)^n \\ & \times [\operatorname{sgn}(\alpha_0) s \cosh(n\beta\mu) f_\rho(mn\beta) - m\beta n \\ & \times \sinh(n\beta\mu) f_{\rho+1}(mn\beta)], \end{aligned} \quad (3.15)$$

and the charge density diverges as  $1/r^{1-2\rho}$ . The vacuum expectation value,  $\langle j^0 \rangle_0$ , diverges in the limit  $r \rightarrow 0$  as  $1/r$  and near the apex it dominates in the total charge density. Though the charge density diverges on the apex, this divergence is integrable and the total charge induced by the planar angle deficit and by the magnetic flux is finite (see below).

Now, let us analyze the limits of low and high temperatures. At low temperatures, for a fixed value of  $mr$ , the main contribution to the finite temperature part of the charge density comes from the  $n = 1$  term in (3.11) and it is given by

$$\begin{aligned} \langle j^0 \rangle \approx & \langle j^0 \rangle_0 + \frac{em}{\pi\beta} \frac{\operatorname{sgn}(\mu)}{e^{\beta(m-|\mu|)}} \left[ \sum_{l=0}^p (-1)^l \right. \\ & \times \cos(\pi l(1/q - 2\operatorname{sgn}(\mu)s\alpha_0)) \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f(q, \operatorname{sgn}(\mu)s\alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} \right], \end{aligned} \quad (3.16)$$

where the  $l = 0$  term corresponds to the Minkowskian part. In this limit one has an exponential suppression of the thermal effects.

To evaluate the charge density at high temperatures, it is convenient to use another representation that is obtained from (3.11) in the way similar to that we have used for (2.39). The new representation has the form

$$\begin{aligned} \langle j^0 \rangle = & \langle j^0 \rangle^{(M)} - \frac{2emT}{\pi} \sum_{n=-\infty}^{\infty} \left\{ s \sum_{l=1}^p (-1)^l s_l \sin(2\pi l\alpha_0) K_0(2rs_l b_n) \right. \\ & \left. - \frac{sq}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) K_0(2rb_n \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right\}, \end{aligned}$$

$$\begin{aligned} & + \frac{\mu - i\pi(2n+1)T}{m} \left[ \sum_{l=1}^p (-1)^l c_l \cos(2\pi l\alpha_0) K_0(2rs_l b_n) \right. \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_1(q, \alpha_0, y) K_0(2rb_n \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right] \Bigg\}, \end{aligned} \quad (3.17)$$

where  $b_n$  is given by (2.40). For a field with zero chemical potential we get

$$\begin{aligned} \langle j^0 \rangle = & -\frac{4semT}{\pi} \sum_{n=0}^{\infty} \left[ \sum_{l=1}^p (-1)^l s_l \sin(2\pi l\alpha_0) K_0(2rs_l b_n^{(0)}) \right. \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) K_0(2rb_n^{(0)} \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right], \end{aligned} \quad (3.18)$$

with the same  $b_n^{(0)}$  as in (2.41).

In the limit of high temperatures, the dominant contributions comes from the terms with  $n = 0$  and  $n = -1$ . For the case  $q > 2$  the leading term corresponds to the  $l = 1$  term and one finds

$$\langle j^0 \rangle = \langle j^0 \rangle^{(M)} - \frac{2ec_1}{\sqrt{rs_1}} \cos(2\pi\alpha_0) \sin(2rs_1\mu) T^{3/2} e^{-2\pi rs_1 T}. \quad (3.19)$$

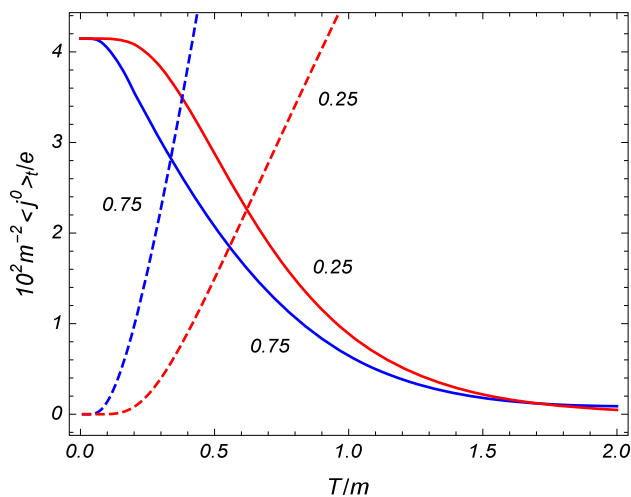
For  $q < 2$  the sums over  $l$  in (3.17) are absent and the dominant contribution to the integrals over  $y$  comes from the region near the lower limit of integration. In this case the effects induced by the conical defect and magnetic flux are suppressed by the factor  $e^{-2\pi rT}$ . In all cases, for  $\mu \neq 0$ , at high temperatures the charge density is dominated by the Minkowskian part  $\langle j^0 \rangle^{(M)}$  with the asymptotic

$$\langle j^0 \rangle^{(M)} \approx \frac{e\mu T}{\pi} \ln 2. \quad (3.20)$$

In the case of zero chemical potential the Minkowskian part vanishes. For points not too close to the cone apex, the contributions induced by the conical defect and by the magnetic flux are exponentially small.

The full curves in Fig. 2 present the dependence of the topological part in the charge density on the temperature. The dashed curves correspond to the charge density in Minkowski spacetime in the absence of the magnetic flux,  $m^{-2} \langle j^0 \rangle^{(M)} / e$ . The numbers near the curves are the values of the ratio  $\mu/m$ . The graphs are plotted for  $q = 1.5$ ,  $\alpha_0 = 0.25$ ,  $mr = 0.5$ , and for the irreducible representation with  $s = 1$ .

In the same way as for the FC, the representation (3.17) also is convenient for the investigation of the charge density at large distances from the origin. As before, the corresponding procedure is similar to that for the high temperature asymptotic. The main contributions come from the terms with  $n = 0$  and  $n = -1$ . Considering  $q > 2$ , one obtains



**Fig. 2** Topological part in the charge density versus the temperature for the irreducible representation with  $s = 1$  (full curves). The graphs are plotted for  $q = 1.5$ ,  $\alpha_0 = 0.25$ ,  $mr = 0.5$ . The dashed curves present the charge density in the Minkowski spacetime. The numbers near the curves are the values of  $\mu/m$

$$\langle j^0 \rangle \approx \langle j^0 \rangle^{(M)} + \frac{2eT}{\sqrt{\pi r s_1}} \times \text{Re} \left[ \frac{s m s_1 \sin(2\pi \alpha_0) + (\mu - i\pi T) c_1 \cos(2\pi \alpha_0)}{b_0^{1/2} e^{2r s_1 b_0}} \right]. \quad (3.21)$$

For  $q < 2$ , the corresponding asymptotic expression is given by

$$\langle j^0 \rangle \approx \langle j^0 \rangle^{(M)} + \frac{e s q m T}{\pi r} \frac{\sin(q\pi \alpha_0)}{\sin(q\pi/2)} \text{Re} \left( \frac{e^{-2rb_0}}{b_0} \right). \quad (3.22)$$

In both cases, the parts induced by the conical defect and the magnetic flux are exponentially small and to the leading order we have  $\langle j^0 \rangle \approx \langle j^0 \rangle^{(M)}$ .

For  $\mu < -m$  ( $\mu > m$ ) the contribution of the particles (antiparticles) to the charge density is still given by (3.7) with the upper (lower) sign. The total charge density is expressed as

$$\begin{aligned} \langle j^0 \rangle = & \langle j^0 \rangle_0 + \langle j^0 \rangle_{\pm} + \frac{e}{\pi} \int_0^\infty dy \frac{\gamma}{e^{\beta(E-|\mu|)} + 1} \\ & \times \left\{ \sum_{l=0}^p (-1)^l \left[ \frac{s m}{E} s_l \sin(2\pi l \alpha_0) \mp c_l \cos(2\pi l \alpha_0) \right] J_0(2\gamma r s_l) \right. \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \frac{\frac{s m}{E} f_2(q, \alpha_0, 2y) \mp f_1(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} J_0(2\gamma r \cosh y) \right\}, \end{aligned} \quad (3.23)$$

where upper and lower signs correspond to  $\mu < -m$  and  $\mu > m$ , respectively, and for  $\langle j^0 \rangle_{\pm}$  one has the expression (3.7). Considering the separate regions  $\gamma < p_0$  and  $\gamma > p_0$ , in the integral corresponding to the second region we can again use the expansion (2.26). The leading terms in the

high temperature asymptotic remain the same as for the case  $|\mu| \leq m$ .

At zero temperature and for the case  $|\mu| > m$ , the only nonzero contribution to the  $\gamma$ -integral in (3.23) comes from the region  $\gamma < p_0$  and one obtains

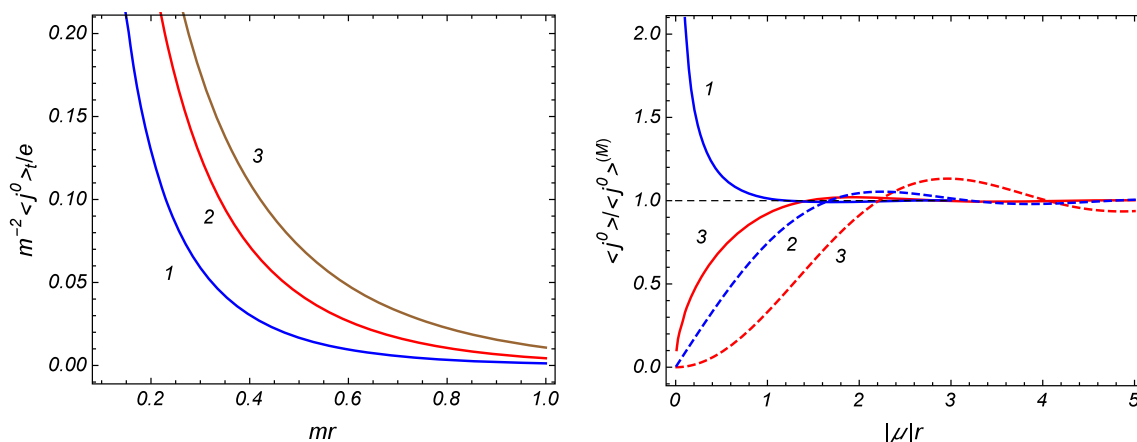
$$\begin{aligned} \langle j^0 \rangle_{T=0} = & \langle j^0 \rangle_0 + \frac{e p_0^2}{\pi} \left\{ \sum_{l=0}^p (-1)^l \times [s s_l \sin(2\pi l \alpha_0) g_1(p_0 r s_l) \right. \\ & \left. + \text{sgn}(\mu) c_l \cos(2\pi l \alpha_0) g_2(p_0 r s_l)] - \frac{q}{\pi} \int_0^\infty dy \right. \\ & \left. \times \frac{\sum_{l=1}^2 s^{l-1} (\text{sgn}(\mu))^l f_l(q, \alpha_0, y) g_{3-l}(p_0 r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right\}, \end{aligned} \quad (3.24)$$

with the functions  $g_1(u)$  and  $g_2(u)$  from (2.49). The appearance of the second term is related to the presence of antiparticles (for  $\mu < -m$ ) or particles (for  $\mu > m$ ) in the states having the energy  $m \leq E \leq |\mu|$ . In the absence of the planar angle deficit and magnetic flux ( $q = 1$ ,  $\alpha = 0$ ), the  $l = 0$  term remains only and for the charge density we obtain the expression:  $\langle j^0 \rangle_{T=0}^{(M)} = \text{sgn}(\mu) e p_0^2 / (2\pi)$ . Subtracting from the right-hand side of (3.24) the term  $l = 0$ , we obtain the charge density induced by the planar angle deficit and by the magnetic flux. At large distances from the cone apex the decay of this part in the charge density is oscillatory with the amplitude decreasing as  $1/r^{3/2}$ . Similar to the case  $|\mu| \leq m$ , the charge density is finite at the apex for  $2|\alpha_0| < 1 - 1/q$  and diverges in the case  $2|\alpha_0| > 1 - 1/q$ . The divergence in the second case is integrable, as  $1/r^{1-2\rho}$ . For a massless field the vacuum charge density is zero and from (3.25) we get the following expression

$$\begin{aligned} \langle j^0 \rangle_{T=0} = & \text{sgn}(\mu) \frac{e \mu^2}{\pi} \left[ \sum_{l=0}^p (-1)^l c_l \cos(2\pi l \alpha_0) g_2(|\mu| r s_l) \right. \\ & \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_1(q, \alpha_0, y) g_2(|\mu| r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right]. \end{aligned} \quad (3.25)$$

In this case the charge density at zero temperature is the same for both the irreducible representations of the Clifford algebra. It is an odd function of the chemical potential and an even function of the magnetic flux.

In Fig. 3 we displayed the charge density as a function of the distance from the cone apex for different values of  $q$  (the numbers near the curves). The left panel presents the topological part in the charge density for the field realizing the irreducible representation with  $s = 1$  and for  $\mu/m = 0.25$ ,  $\alpha_0 = 0.25$ ,  $T/m = 0.5$ . In the right panel the ratio  $\langle j^0 \rangle / \langle j^0 \rangle^{(M)}$  is plotted at zero temperature for a massless fermionic field. The full and dashed curves correspond to  $\alpha_0 = 0.25$  and  $\alpha_0 = 0$ , respectively. Note that for  $\alpha_0 = 0.25$  and  $q = 2$  one has  $\langle j^0 \rangle_{T=0} = \langle j^0 \rangle_{T=0}^{(M)}$ .



**Fig. 3** Charge density as a function of the distance from the apex for different values of the parameter  $q$  (the numbers near the graphs). On the left panel, the topological part is presented for the representation  $s = 1$  and for  $\mu/m = 0.25$ ,  $\alpha_0 = 0.25$ ,  $T/m = 0.5$ . The right panel

gives the dependence of the ratio  $\langle j^0 \rangle / \langle j^0 \rangle^{(M)}$  at zero temperature. The full and dashed curves on this panel correspond to  $\alpha_0 = 0.25$  and  $\alpha_0 = 0$

Let us denote by  $\Delta Q$  the total charge induced by the planar angle deficit and by the magnetic flux:

$$\Delta Q = \int_0^\infty dr r \int_0^{\phi_0} d\phi \left[ \langle j^0 \rangle - \langle j^0 \rangle^{(M)} \right]. \quad (3.26)$$

In the case  $|\mu| \leq m$  we use the representation (3.11). The integration over the radial coordinate is done with the help of the formula

$$\int_0^\infty dr r f_v(\sqrt{a^2 + b^2 r^2}) = f_{v-1}(a)/b^2. \quad (3.27)$$

After the summation over  $n$  we get

$$\begin{aligned} \Delta Q = \Delta Q_0 + \frac{e}{2} \sum_{\delta=\pm 1} \frac{\delta}{e^{\beta(m-\delta\mu)} + 1} & \left[ \frac{1}{q} \sum_{l=1}^p \frac{(-1)^l}{s_l^2} \right. \\ & \times \cos(\pi l (1/q - 2s\delta\alpha_0)) \\ & \left. - \frac{1}{\pi} \int_0^\infty dy \frac{f_2(q, s\delta\alpha_0, y) \cosh^{-2} y}{\cosh(2qy) - \cos(q\pi)} \right], \end{aligned} \quad (3.28)$$

where

$$\begin{aligned} \Delta Q_0 = -\frac{se}{2q} & \left[ \sum_{l=1}^p \frac{(-1)^l}{s_l} \sin(2\pi l\alpha_0) - \frac{q}{\pi} \right. \\ & \left. \times \int_0^\infty dy \frac{f_2(q, \alpha_0, y) \cosh^{-2} y}{\cosh(2qy) - \cos(q\pi)} \right], \end{aligned} \quad (3.29)$$

is the vacuum charge and the second term in the right-hand side presents the contribution from particles and antiparticles. These expressions are further simplified by using the relations (A.4) and (A.6):

$$\begin{aligned} \Delta Q = \Delta Q_0 + \frac{e}{2q} \sum_{\delta=\pm 1} \frac{\delta}{e^{\beta(m-\delta\mu)} + 1} \\ \times \left[ \frac{1-q^2}{12} + q\alpha_0 (q\alpha_0 - s\delta) \right], \end{aligned} \quad (3.30)$$

with  $\Delta Q_0 = se\alpha_0/2$ . For  $|\mu| < m$ , in the zero temperature limit one has  $\lim_{T \rightarrow 0} \Delta Q = \Delta Q_0$  and the topological part of the charge coincides with that for the vacuum state.

In the case  $|\mu| > m$ , the charge at zero temperature differs from the vacuum charge  $\Delta Q_0$ . It is obtained by the integration of the right-hand side of (3.24), omitting the term  $l = 0$  (the Minkowskian part). By taking into account that

$$\int_0^\infty dy y g_l(ay) = \frac{1}{4a^2}, \quad l = 1, 2, \quad (3.31)$$

and using the relation (A.4), one finds the following expression

$$\begin{aligned} \Delta Q_{T=0} = \Delta Q_0 + \text{sgn}(\mu) \frac{e}{2q} \\ \times \left\{ \frac{1-q^2}{12} + q\alpha_0 [q\alpha_0 - s\text{sgn}(\mu)] \right\}, \end{aligned} \quad (3.32)$$

with the same  $\Delta Q_0$  as in (3.30). For a given sign of the chemical potential, the zero temperature charge is completely determined by the topological parameters of the model. Note that in the evaluation of the integral (3.31) with the function  $g_1(u)$  from (2.49) one cannot change the order of integrations. In order to escape this difficulty, we introduce in the integrand the function  $e^{-by}$ ,  $b > 0$ . With this function, changing the integrations order, the integral over  $y$  involving the Bessel function is evaluated by using the formula from [71]. Then, after the evaluation of the integral over  $x$ , we take the limit  $b \rightarrow 0$ . For  $|\mu| = m$  the expression (3.32) coincides with that obtained from (3.30) in the limit  $T \rightarrow 0$ .

#### 4 Current density

In this section we consider the expectation value of the spatial components for the current density. First of all we can see that the current density along the radial direction is zero,  $\langle j^1 \rangle = 0$ , and the only nonzero component is along the azimuthal direction. The contributions to the azimuthal current coming from the particles and antiparticles are obtained from (3.3) by using the mode functions (2.9). For the physical component,  $\langle j_\phi \rangle$ , connected to the contravariant one by the relation  $\langle j_\phi \rangle = r \langle j^2 \rangle$ , one gets the following expression:

$$\langle j_\phi \rangle_\pm = \pm \frac{qe}{2\pi} \sum_j \epsilon_j \int_0^\infty d\gamma \frac{\gamma^2}{E} \frac{J_{\beta_j}(\gamma r) J_{\beta_j + \epsilon_j}(\gamma r)}{e^{\beta(E \mp \mu)} + 1}. \quad (4.1)$$

Note that the expectation value of the azimuthal current density does not depend on the parameter  $s$  in (2.2) and it is the same for both the irreducible representations of the Clifford algebra.

By making use of the recurrence relation for the Bessel function we can show that

$$\epsilon_j J_{\beta_j}(x) J_{\beta_j + \epsilon_j}(x) = \frac{1}{x} \left( \epsilon_j \beta_j - \frac{1}{2} x \partial_x \right) J_{\beta_j}^2(x). \quad (4.2)$$

Substituting this into (4.1) and using the representation (2.14), the contributions to the azimuthal current from the particles and antiparticles are presented as

$$\begin{aligned} \langle j_\phi \rangle_\pm &= \frac{eqr}{2\pi} \int_0^\infty d\gamma \frac{\gamma^3/E}{e^{\beta(E \mp \mu)} + 1} \frac{1}{2\pi i} \\ &\times \int_{c-i\infty}^{c+i\infty} \frac{dt}{t^2} e^{t/2 - \gamma^2 r^2/t} \sum_{\delta=\pm 1} \delta \mathcal{I}(q, \delta\alpha_0, \gamma^2 r^2/t). \end{aligned} \quad (4.3)$$

By taking into account (2.17), the integral over  $t$  is expressed in terms of the Bessel function of the order 1 and we get

$$\begin{aligned} \langle j_\phi \rangle_\pm &= \frac{e}{\pi} \int_0^\infty d\gamma \frac{\gamma^2/E}{e^{\beta(E \mp \mu)} + 1} \\ &\times \left\{ \sum_{l=1}^p (-1)^l \sin(2\pi l \alpha_0) J_1(2\gamma r s_l) \right. \\ &\left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) J_1(2\gamma r \cosh y)}{[\cosh(2qy) - \cos(q\pi)] \cosh y} \right\}. \end{aligned} \quad (4.4)$$

As is seen, the azimuthal current is an odd function of the parameter  $\alpha_0$ . Note that for the zero chemical potential,  $\mu = 0$ , the particles and antiparticles give the same contributions to the total current density. As before, we will consider the cases  $|\mu| \leq m$  and  $|\mu| > m$  separately.

For  $|\mu| \leq m$ , by using the expansion (2.26), one finds

$$\begin{aligned} \langle j_\phi \rangle_\pm &= -\frac{2^{3/2} em^3 r}{\pi^{3/2}} \sum_{n=1}^\infty (-1)^n e^{\pm n\beta\mu} \left[ \sum_{l=1}^p (-1)^l s_l \sin(2\pi l \alpha_0) \right. \\ &\times f_{3/2}(c_{nl}) - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \Big]. \end{aligned} \quad (4.5)$$

For the total current density this gives

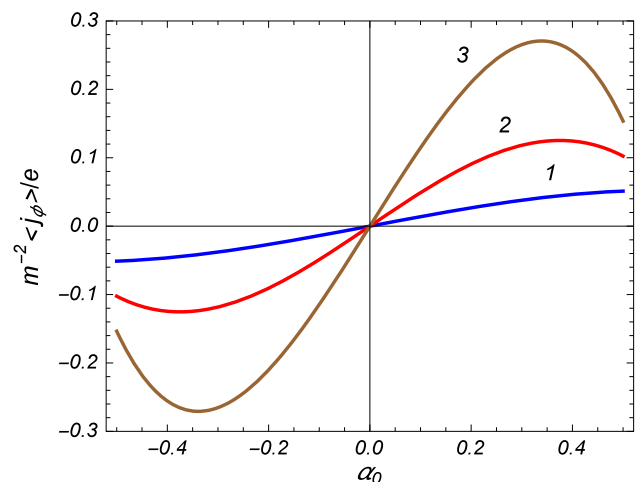
$$\begin{aligned} \langle j_\phi \rangle &= -\frac{2^{5/2} em^3 r}{\pi^{3/2}} \sum_{n=0}^\infty (-1)^n \cosh(n\beta\mu) \left[ \sum_{l=1}^p (-1)^l f_{3/2}(c_{nl}) \right. \\ &\times s_l \sin(2\pi l \alpha_0) - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \Big], \end{aligned} \quad (4.6)$$

where the  $n = 0$  term corresponds to the vacuum expectation value [50]:

$$\begin{aligned} \langle j_\phi \rangle_0 &= -\frac{2^{3/2} em^3 r}{\pi^{3/2}} \left[ \sum_{l=1}^p (-1)^l s_l \sin(2\pi l \alpha_0) f_{3/2}(2mr s_l) \right. \\ &\left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) f_{3/2}(2mr \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right]. \end{aligned} \quad (4.7)$$

Unlike to the FC and charge density, the azimuthal current density has definite parity with respect to the reflections  $\alpha_0 \rightarrow -\alpha_0$  and  $\mu \rightarrow -\mu$ : it is an odd function of  $\alpha_0$  and an even function of the chemical potential. For an integer  $q$  and for special values of  $\alpha_0$ , given by (2.25), the integral terms in (4.6) and (4.7) vanish.

In Fig. 4, the azimuthal current density is plotted versus the parameter  $\alpha_0$  for  $\mu/m = 0.25$ ,  $mr = 0.5$ ,  $T/m = 0.5$ .



**Fig. 4** Azimuthal current density as a function of  $\alpha_0$  for separate values of the parameter  $q$  (numbers near the curves) and for  $\mu/m = 0.25$ ,  $mr = 0.5$ ,  $T/m = 0.5$



The numbers near the curves are the values of  $q$ . As we have mentioned, the current density is an odd function of  $\alpha_0$ .

For the massless field, due to the condition  $\mu \leq m$ , we also should take  $\mu = 0$ . Using the asymptotic expression for the Macdonald function for small values of the argument, we find

$$\langle j_\phi \rangle = -\frac{4er}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[ \sum_{l=1}^p \frac{(-1)^l s_l \sin(2\pi l \alpha_0)}{(n^2 \beta^2 + 4s_l^2 r^2)^{3/2}} - \frac{q}{\pi} \int_0^\infty dy \frac{(n^2 \beta^2 + 4r^2 \cosh^2 y)^{-3/2}}{\cosh(2qy) - \cos(q\pi)} f_2(q, \alpha_0, y) \right]. \quad (4.8)$$

In the case of the Minkowski bulk with a magnetic flux, the current density is obtained from (4.6) with  $q = 1$ :

$$\langle j_\phi \rangle = \frac{2^{5/2} e m^3 r}{\pi^{5/2}} \sin(\pi \alpha_0) \sum_{n=0}^{\infty} (-1)^n \cosh(n\beta\mu) \times \int_0^\infty dy \cosh(2\alpha_0 y) f_{3/2}(c_n(y)). \quad (4.9)$$

At the cone apex, the thermal part in the azimuthal current density vanishes as  $r$  for  $2|\alpha_0| < 1 - 1/q$  and as  $r^{2\rho}$  for  $2|\alpha_0| > 1 - 1/q$ . In the first case the leading term is obtained directly from the  $n \neq 0$  term in (4.6) putting  $r = 0$ . This is reduced to the substitutions  $c_{nl} = c_n(y) = mn\beta$ . In the second case, the leading term in the asymptotic expansion of the thermal part is found in the way similar to that we have used for the FC and has the form

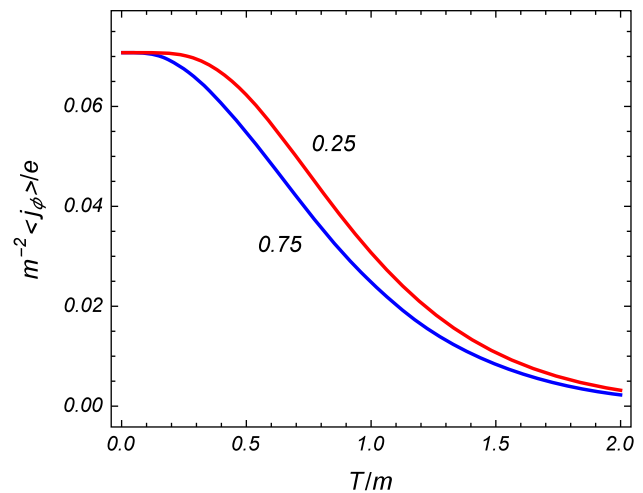
$$\langle j_\phi \rangle \approx \langle j_\phi \rangle_0 + \frac{em^2 q (mr)^{2\rho}}{2^{\rho-1} \pi^{5/2}} \Gamma(1/2 - \rho) \operatorname{sgn}(\alpha_0) \cos(\pi\rho) \times \sum_{n=0}^{\infty} (-1)^n \cosh(n\beta\mu) f_{\rho+1}(mn\beta). \quad (4.10)$$

The vacuum expectation value diverges as  $1/r^2$  and it dominates near the apex.

Considering the limit of low temperatures for a fixed value of  $mr$ , the main contribution for the thermal part of the azimuthal current comes from the  $n = 1$  term and the leading term is given by

$$\langle j_\phi \rangle \approx \langle j_\phi \rangle_0 + \frac{2emT^2 r}{\pi e^{\beta(m-|\mu|)}} \left[ \sum_{l=1}^p (-1)^l s_l \sin(2\pi l \alpha_0) - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y)}{\cosh(2qy) - \cos(q\pi)} \right]. \quad (4.11)$$

In this limit the contribution of the finite temperature effects is suppressed by the factor  $e^{-\beta(m-|\mu|)}$ .



**Fig. 5** Azimuthal current density as a function of temperature. The numbers near the curves are the values of the ratio  $\mu/m$  and the graphs are plotted for  $q = 1.5$ ,  $\alpha_0 = 0.25$ ,  $mr = 0.5$

An alternative representation for the current density in the case  $|\mu| \leq m$  is obtained by using the formula (2.38):

$$\langle j_\phi \rangle = -\frac{2eT}{\pi} \sum_{n=-\infty}^{+\infty} b_n \left\{ \sum_{l=1}^p (-1)^l \sin(2\pi l \alpha_0) K_1(2r s_l b_n) - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) K_1(2r b_n \cosh y)}{[\cosh(2qy) - \cos(q\pi)] \cosh y} \right\}, \quad (4.12)$$

where  $b_n$  is given by (2.40).

At high temperatures, assuming that  $rT \gg 1$ , the dominant contribution in (4.12) comes from the terms  $n = 0$  and  $n = -1$ . For  $q > 2$  one finds

$$\langle j_\phi \rangle \approx \frac{2eT^{3/2}}{\sqrt{s_1} r} \sin(2\pi \alpha_0) \frac{\cos(2r s_1 \mu)}{e^{2\pi r s_1 T}}. \quad (4.13)$$

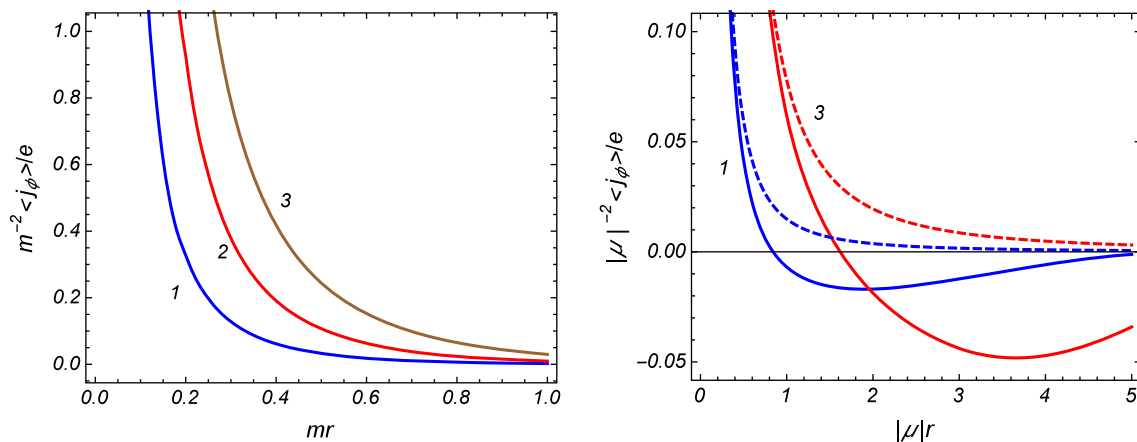
In the case  $q < 2$ , assuming that  $T \gg 1/[\pi r \sin^2(q\pi/2)]$ , the leading term is given by

$$\langle j_\phi \rangle \approx \frac{eqT \sin(\pi q \alpha_0) \cos(2r\mu)}{\pi r \sin(\pi q/2) e^{2\pi r T}}. \quad (4.14)$$

The dependence of the current density on temperature is plotted in Fig. 5 for two values of the ratio  $\mu/m$  (numbers near the graphs). The graphs are plotted for  $q = 1.5$ ,  $\alpha_0 = 0.25$ ,  $mr = 0.5$ .

For the investigation of the asymptotic of the azimuthal current at large distances from the origin we again use the representation (4.12). In the case  $q > 2$ , the dominant contribution comes from the terms with  $l = 1$ ,  $n = 0, 1$  and we get

$$\langle j_\phi \rangle \approx \frac{2eT \sin(2\pi \alpha_0)}{\sqrt{s_1} r} \operatorname{Re} \left( b_0^{1/2} e^{-2r s_1 b_0} \right). \quad (4.15)$$



**Fig. 6** Current density versus the distance from the apex for  $\alpha_0 = 0.25$  and for different values of the parameter  $q$  (the numbers near the graphs). The graphs on the left panel are plotted for  $\mu/m = 0.25$ ,

$T/m = 0.5$ . The right panel presents the zero temperature (full curves) and vacuum currents (dashed curves) for a massless field

For  $q < 2$  and not too close to 2, the leading term is given by the expression

$$\langle j_\phi \rangle \approx \frac{eq \sin(\pi q \alpha_0) T}{\pi \sin(\pi q/2) r} \operatorname{Re} \left( e^{-2rb_0} \right). \quad (4.16)$$

Now let us consider the current density for the case  $|\mu| > m$ . The corresponding expression has the form

$$\begin{aligned} \langle j_\phi \rangle &= \langle j_\phi \rangle_0 + \langle j_\phi \rangle_\pm + \frac{e}{\pi} \int_0^\infty dy \frac{\gamma^2/E}{e^{\beta(E-|\mu|)} + 1} \\ &\times \left\{ \sum_{l=1}^p (-1)^l \sin(2\pi l \alpha_0) \right. \\ &\times J_1(2\gamma r s_l) - \frac{q}{\pi} \int_0^\infty dy \\ &\times \left. \frac{f_2(q, \alpha_0, y) J_1(2\gamma r \cosh y)}{[\cosh(2qy) - \cos(q\pi)] \cosh y} \right\}, \end{aligned} \quad (4.17)$$

where the upper and lower signs correspond to the cases  $\mu < -m$  and  $\mu > m$ , respectively. The expression for  $\langle j_\phi \rangle_\pm$  in the right-hand side is given by (4.5). At zero temperature,  $T \rightarrow 0$ , we get

$$\begin{aligned} \langle j_\phi \rangle_{T=0} &= \langle j_\phi \rangle_0 - \frac{ep_0^3}{2\pi m} \left\{ \sum_{l=1}^p (-1)^l \sin(2\pi l \alpha_0) g'_1(p_0 r s_l) \right. \\ &\times \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) g'_1(p_0 r \cosh y)}{[\cosh(2qy) - \cos(q\pi)] \cosh y} \right\}, \end{aligned} \quad (4.18)$$

with the function  $g_1(u)$  from (2.49) and  $g'_1(u) = \partial_u g_1(u)$ . The second term in the right-hand side is the contribution from the antiparticles for  $\mu < -m$  and from the particles for  $\mu > m$ . In the case of a massless field we get

$$\begin{aligned} \langle j_\phi \rangle_{T=0} &= \frac{e}{4\pi r^2} \\ &\times \left\{ \sum_{l=1}^p \frac{(-1)^l}{s_l^2} \sin(2\pi l \alpha_0) g_0(|\mu| r s_l) \right. \\ &\times \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y) g_0(|\mu| r \cosh y)}{[\cosh(2qy) - \cos(q\pi)] \cosh^3 y} \right\}, \end{aligned} \quad (4.19)$$

with the function

$$g_0(u) = \int_0^{2u} dx x J_1(x) - 1. \quad (4.20)$$

The part in (4.19) coming from  $-1$  in the right-hand side of (4.20) corresponds to the vacuum current density. It depends on the radial coordinate as  $1/r^2$ .

The dependence of the azimuthal current density on the distance from the cone apex is displayed in Fig. 6 for  $\alpha_0 = 0.25$  and for separate values of  $q$  (numbers near the curves). The graphs on the left panel are plotted for  $\mu/m = 0.25$ ,  $T/m = 0.5$ . The full curves on the right panel present the current density at zero temperature for a massless fermionic field with the chemical potential  $\mu$ . The dashed curves are the vacuum current densities in the same model.

## 5 Expectation values in parity and time-reversal symmetric models

In the discussion above we have considered a fermionic field realizing the irreducible representation of the Clifford algebra. In this representation, the mass term in the Lagrangian density is not invariant under the parity and time-reversal transformations ( $P$ - and  $T$ -transformations). In order to recover the  $P$ - and  $T$ -invariance, we note that in  $(2+1)$ -

dimensions the  $\gamma^2$  matrix can be represented in two different ways:  $\gamma^2 = \gamma_{(s)}^2 = -is\sqrt{-g^{22}}\gamma^0\gamma^1$ , with  $s = \pm 1$ . As a consequence, the Clifford algebra has two inequivalent representations corresponding to the upper and lower signs. Our choice in (2.3) corresponds to the representation with the upper sign. Consider two two-component spinor fields,  $\psi_{(+1)}$  and  $\psi_{(-1)}$ , with the combined Lagrangian density  $\mathcal{L} = \sum_{s=\pm 1} \bar{\psi}_{(s)}(i\gamma_{(s)}^\mu D_\mu - m)\psi_{(s)}$ , where  $\gamma_{(s)}^\mu = (\gamma^0, \gamma^1, \gamma_{(s)}^2)$ . We can see that, by suitable transformations of the fields (see, for example [72, 73]), this Lagrangian is invariant under  $P$ - and  $T$ -transformations (in the absence of magnetic fields). Defining new fields  $\psi'_{(+1)} = \psi_{(+1)}$ ,  $\psi'_{(-1)} = \gamma^0\gamma^1\psi_{(-1)}$ , the Lagrangian density is transformed to the form

$$\mathcal{L} = \sum_{s=\pm 1} \bar{\psi}'_{(s)}(i\gamma^\mu D_\mu - sm)\psi'_{(s)}, \quad (5.1)$$

with  $\gamma^\mu = (\gamma^0, \gamma^1, \gamma_{(+1)}^2)$ . From here it follows that the field  $\psi'_{(-1)}$  satisfies the same equation as  $\psi_{(+1)}$  with the opposite sign for the mass term. We can write the Lagrangian (5.1) in terms of the four-component spinor  $\Psi = (\psi'_{(+1)}, \psi'_{(-1)})^T$  as  $\mathcal{L} = \bar{\Psi}(i\gamma_{(1)}^\mu D_\mu - m\eta)\Psi$  with  $4 \times 4$  matrices  $\gamma_{(1)}^\mu = I_2 \otimes \gamma^\mu$  and  $\eta = \sigma_3 \otimes I_2$ , where  $I_2 = \text{diag}(1, 1)$ . Another form is obtained by using the  $4 \times 4$  reducible representation of gamma matrices  $\gamma_{(2)}^\mu = \sigma_3 \otimes \gamma^\mu$  with the Lagrangian density  $\mathcal{L} = \bar{\Psi}(i\gamma_{(2)}^\mu D_\mu - m)\Psi$ .

As is seen from (5.1), the expectation values in the corresponding models are obtained from the formulas given above by summing the contributions from the fields  $\psi_{(+1)}$  and  $\psi_{(-1)}$ . For the first one the expressions are obtained from those presented in the previous sections taking  $s = 1$ . In order to find the contribution of the field  $\psi_{(-1)}$ , we note that  $\bar{\psi}_{(-1)}\psi_{(-1)} = -\bar{\psi}'_{(-1)}\psi'_{(-1)}$  and  $\bar{\psi}_{(-1)}\gamma_{(-1)}^\mu\psi_{(-1)} = \bar{\psi}'_{(-1)}\gamma^\mu\psi'_{(-1)}$ , with  $\mu = 0, 2$ . The expectation values  $\langle \bar{\psi}'_{(-1)}\psi'_{(-1)} \rangle$  and  $\langle \bar{\psi}'_{(-1)}\gamma^\mu\psi'_{(-1)} \rangle$  are obtained from the formulas in the previous sections taking  $s = -1$ . As a result, for the total expectation values we get

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= \sum_{s=\pm 1} \langle \bar{\psi}_{(s)}\psi_{(s)} \rangle = \sum_{s=\pm 1} s \langle \bar{\psi}'_{(s)}\psi'_{(s)} \rangle, \\ \langle \bar{\psi}\gamma^\mu\psi \rangle &= \sum_{s=\pm 1} \langle \bar{\psi}_{(s)}\gamma_{(s)}^\mu\psi_{(s)} \rangle = \sum_{s=\pm 1} \langle \bar{\psi}'_{(s)}\gamma^\mu\psi'_{(s)} \rangle. \end{aligned} \quad (5.2)$$

Here we give the final expressions. First of all, note that the contributions to the azimuthal current density from the fields  $\psi_{(+1)}$  and  $\psi_{(-1)}$  coincide and the expression for the total current is obtained from (4.6) with the additional coefficient 2.

For the FC, summing the separate contributions in accordance with (5.2), in the case  $|\mu| \leq m$  one has

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= \langle \bar{\psi}\psi \rangle^{(M)} - \frac{2^{5/2}m^2}{\pi^{3/2}} \sum_{n=0}^{\infty} (-1)^n \cosh(n\beta\mu) \\ &\times \left[ \sum_{l=1}^p (-1)^l c_l \cos(2\pi l\alpha_0) f_{1/2}(c_{nl}) - \frac{q}{\pi} \int_0^\infty dy \right. \\ &\times \left. \frac{f_1(q, \alpha_0, y) f_{1/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right], \end{aligned} \quad (5.3)$$

where the contribution  $\langle \bar{\psi}\psi \rangle^{(M)}$  is given by (2.32) with the additional coefficient 2 and taking  $s = 1$ . The FC in the vacuum state,  $\langle \bar{\psi}\psi \rangle_0$ , corresponds to the  $n = 0$  term in (5.3). Now, the FC is an even function of the chemical potential  $\mu$  and of the magnetic flux parameter  $\alpha_0$ . For  $|\mu| > m$ , the FC at zero temperature contains two contributions:

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_{T=0} &= \langle \bar{\psi}\psi \rangle_0 + \frac{2p_0^2}{\pi} \left[ \sum_{l=0}^p (-1)^l c_l \cos(2\pi l\alpha_0) g_1(p_0 r s l) \right. \\ &\times \left. - \frac{q}{\pi} \int_0^\infty dy \frac{f_1(q, \alpha_0, y) g_1(p_0 r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right]. \end{aligned} \quad (5.4)$$

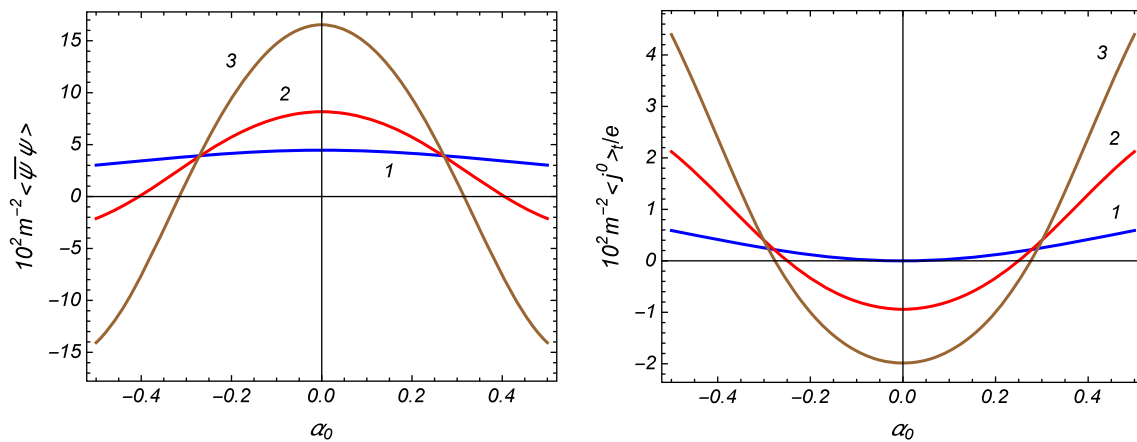
The second term comes from the particles for  $\mu > 0$  and antiparticles for  $\mu < 0$ . It is symmetric under the change of the sign for the chemical potential,  $\mu \rightarrow -\mu$ .

In the left panel of Fig. 7 we have plotted the FC, defined by (5.3), as a function of  $\alpha_0$  for different values of  $q$  (the numbers near the curves). The values of the other parameters are the same as in Fig. 1. As is seen, in the absence of the planar angle deficit ( $q = 1$ ) the FC is positive. With the increasing  $q$ , started from some critical value of the planar angle deficit, the nodes of the FC appear at which the condensate vanishes. The corresponding values for  $|\alpha_0|$  decrease with increasing  $q$ . Note that the nonzero FC is closely related to the symmetry breaking in field-theoretical models and its vanishing leads to the symmetry restoration.

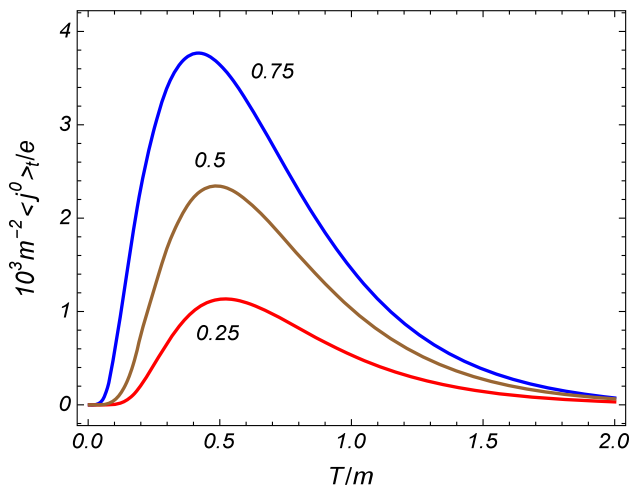
For the charge density in  $P$ - and  $T$ -symmetric models, in the case  $|\mu| \leq m$ , we get

$$\begin{aligned} \langle j^0 \rangle &= \langle j^0 \rangle^{(M)} - \frac{2^{5/2}em^3}{\pi^{3/2}T} \sum_{n=1}^{\infty} (-1)^n n \sinh(n\beta\mu) \\ &\times \left[ \sum_{l=1}^p (-1)^l c_l \cos(2\pi l\alpha_0) f_{3/2}(c_{nl}) - \frac{q}{\pi} \int_0^\infty dy \right. \\ &\times \left. \frac{f_1(q, \alpha_0, y) f_{3/2}(c_n(y))}{\cosh(2qy) - \cos(q\pi)} \right], \end{aligned} \quad (5.5)$$

with the Minkowskian part from (3.12) with the additional coefficient 2. Note that the vacuum charge density vanishes. The charge density is an even function of  $\alpha_0$  and an odd function of the chemical potential. In particular, it vanishes for the zero chemical potential. This is because of the cancellation of the contributions coming from the separate irreducible representations. In the right panel of Fig. 7, the topological part



**Fig. 7** Fermion condensate (left panel) and the charge density (right panel) in a  $P$ - and  $T$ -invariant model with two spinors  $\psi_{(+1)}$  and  $\psi_{(-1)}$  as functions of the parameter  $\alpha_0$ . The values of the other parameters are the same as in Fig. 1



**Fig. 8** The same as in Fig. 2 in a  $P$ - and  $T$ -invariant model. The numbers near the curves are the values of the ratio  $\mu/m$

in the charge density is displayed as a function of  $\alpha_0$  in the  $P$ - and  $T$ -invariant model with two spinors  $\psi_{(+1)}$  and  $\psi_{(-1)}$ . The numbers near the curves are the values of the parameter  $q$  and we have taken  $\mu/m = 0.25$ ,  $mr = 0.5$ ,  $T/m = 0.5$ .

The dependence of the topological part in the charge density on the temperature is presented in Fig. 8 for different values of the ratio  $\mu/m$  (numbers near the curves). For the other parameters we have taken the same values as in Fig. 2.

In the case  $|\mu| > m$ , the charge density is obtained by summing the expressions in the right-hand side of (3.23) for  $s = 1$  and  $s = -1$ . In particular, at zero temperature we have

$$\langle j^0 \rangle_{T=0} = \frac{2}{\pi} \text{sgn}(\mu) e p_0^2 \left[ \sum_{l=0}^p (-1)^l c_l \cos(2\pi l \alpha_0) g_2(p_0 r s_l) - \frac{q}{\pi} \int_0^\infty dy \frac{f_1(q, \alpha_0, y) g_2(p_0 r \cosh y)}{\cosh(2qy) - \cos(q\pi)} \right]. \quad (5.6)$$

The zero temperature charge density comes from the particles or antiparticles for the cases  $\mu > 0$  and  $\mu < 0$  respectively.

In Fig. 9 the dependence of the charge density on the radial coordinate is plotted for different values of  $q$  (numbers near the graphs). For the left panel we have taken  $\mu/m = 0.25$ ,  $\alpha_0 = 0.25$ ,  $T/m = 0.5$ . Note that for these values of the parameters and in the case  $q = 2$  the topological part vanishes. On the right panel the ratio  $\langle j^0 \rangle / \langle j^0 \rangle^{(M)}$  is plotted at zero temperature. The full and dashed curves correspond to  $\alpha_0 = 0.25$  and  $\alpha_0 = 0$ , respectively. For  $\alpha_0 = 0.25$  and  $q = 2$  one has  $\langle j^0 \rangle = \langle j^0 \rangle^{(M)}$ .

For the chemical potential in the range  $|\mu| \leq m$ , the total charge induced by the planar angle deficit and by the magnetic flux is presented as

$$\Delta Q = eq \left( \frac{1/q^2 - 1}{12} + \alpha_0^2 \right) \sum_{\delta=\pm 1} \frac{\delta}{e^{\beta(m-\delta\mu)} + 1}. \quad (5.7)$$

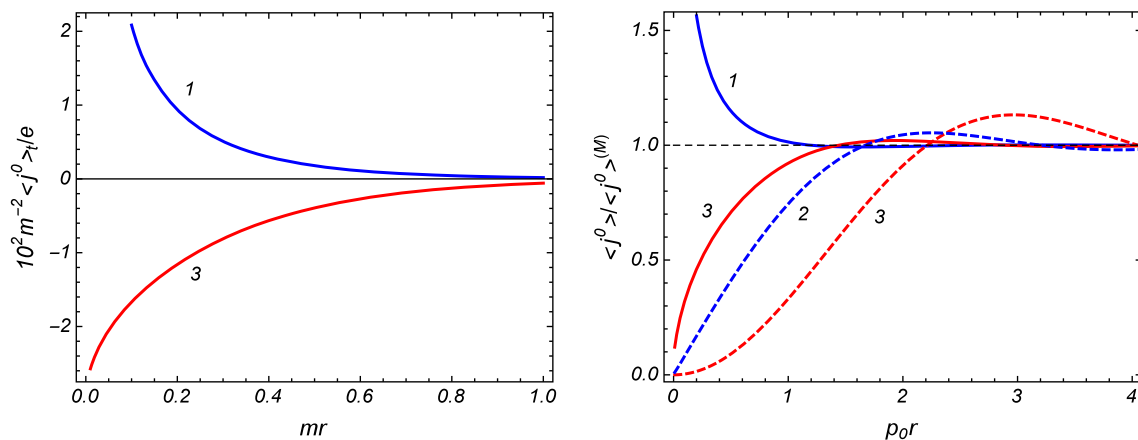
In the case of zero chemical potential the charge vanishes. For  $|\mu| \geq m$ , for the topological part in the charge at zero temperature one gets

$$\Delta Q_{T=0} = \text{sgn}(\mu) eq \left( \frac{1/q^2 - 1}{12} + \alpha_0^2 \right). \quad (5.8)$$

This charge is completely determined by the topological parameters of the model.

In summing the contributions from the spinors  $\psi_{(+1)}$  and  $\psi_{(-1)}$  we have assumed that the parameter  $\alpha$  is the same for both these spinors. However, in general, this parameter can be different for separate spinors. In particular, this difference can be a consequence of different phases in the quasiperiodicity condition (2.51) along the azimuthal direction.

Among the most important physical systems, with the low-energy sector described by a Dirac-like theory in two spatial dimensions, is graphene. The long-wavelength excitations of the electronic subsystem in graphene are described by a pair



**Fig. 9** Charge density in the  $P$ - and  $T$ -invariant model as a function of the distance from the apex for separate values of  $q$  (the numbers near the curves). On the left panel, the topological part is presented

of two-component spinors  $\psi_+$  and  $\psi_-$ , corresponding to the two different inequivalent points  $\mathbf{K}_+$  and  $\mathbf{K}_-$  at the corners of the two-dimensional Brillouin zone (see [4,5]). The separate components of these spinors give the amplitude of the wave function on the  $A$  and  $B$  triangular sublattices of the graphene hexagonal lattice. Graphitic cones are obtained from planar graphene sheet if one or more sectors with the angle  $\pi/3$  are excised and the remainder is joined. The opening angle of the cone is connected to the number of the removed sectors,  $N_c$ , by the relation  $\phi_0 = 2\pi(1 - N_c/6)$ , with  $N_c = 1, 2, \dots, 5$ . All these angles have been observed in experiments [74,75]. At the apex of the graphitic cone the hexagon of the planar graphene lattice is replaced by a polygon having  $6 - N_c$  sites. The periodicity conditions for the combined bispinor  $\Psi = (\psi_+, \psi_-)$  under the rotation around the cone apex are discussed in [76–79]. For even values of  $N_c$  these conditions do not mix the spinors  $\psi_+$  and  $\psi_-$ , and we can apply the formulas given above.

Note that the gap in the energy spectrum,  $\Delta$ , is connected with the Dirac mass  $m$  by the relation  $\Delta = mv_F^2$ , where  $v_F$  is the Fermi velocity (for graphene  $v_F \approx 10^8$  cm/s). It is essential in many physical applications of the model. In planar condensed matter systems the energy gap can be generated by a number of mechanisms (see, for example, [4,5,46–49,80–84] and references therein). The latter include the breaking of symmetry between the sublattices by introducing a staggered on-site energy, the deformations of bonds in the lattice, fermion contact interactions and the statistical gauge interactions. Another approach is to attach a planar condensed matter system to a substrate, with an interaction breaking the sublattice symmetry. The energy gap can also be generated by the compactification of spatial dimensions (examples are graphene-made cylindrical and toroidal nanotubes). Some mechanisms for the gap generation give rise to mass terms

for  $\mu/m = 0.25$ ,  $\alpha_0 = 0.25$ ,  $T/m = 0.5$ . The right panel displays the ratio  $\langle j^0 \rangle / \langle j^0 \rangle^{(M)}$  at zero temperature. The full and dashed curves correspond to  $\alpha_0 = 0.25$  and  $\alpha_0 = 0$

with a matrix structure different from that we have considered here (for a review of possible mass terms in graphene see [4,5]).

## 6 Conclusion

We have investigated the expectation values of the FC, charge and current densities for a massive fermion field with nonzero chemical potential in thermal equilibrium on the background of  $(2+1)$ -dimensional conical spacetime with an arbitrary planar angle deficit in the presence of a magnetic flux located at the cone apex. For both the spinor fields realizing the two inequivalent representations of the Clifford algebra, the expectation values are decomposed into three contributions coming from the vacuum expectation values, from the particles and from the antiparticles. All these contributions are periodic functions of the magnetic flux with the period equal to the flux quantum. The vacuum expectation values have been investigated earlier and here we are mainly concerned with the finite temperature effects.

In the case  $|\mu| \leq m$ , the FC is presented in the form (2.31), where  $s = 1$  and  $s = -1$  correspond to two irreducible representations of the Clifford algebra. With these representations, the mass term breaks the  $P$ - and  $T$ -invariances and, related to this, the FC has no definite parity with respect to the reflections  $\alpha_0 \rightarrow -\alpha_0$  and  $\mu \rightarrow -\mu$ . For a massless field with the zero chemical potential the FC vanishes. In the massive case, for integer values of the parameter  $q$  and for special values of the magnetic flux given by (2.25), the integral terms in (2.31) vanish. In the absence of the magnetic flux, the FC is given by (2.34) and it has opposite signs for two irreducible representations with the same modulus. Another special case corresponds to Minkowski bulk in the presence of a mag-



netic flux ( $q = 1$ ) with the FC given in (2.35). In order to clarify the behavior of the FC, we have considered different asymptotics of the general formula. The finite temperature part in the FC is finite on the apex for  $2|\alpha_0| < 1 - 1/q$  and diverges as  $1/r^{1-2\rho}$ , with  $\rho = q(1/2 - |\alpha_0|)$ , in the case  $2|\alpha_0| > 1 - 1/q$ . The divergence is related to the irregular mode. For a massive field, the vacuum FC diverges on the apex as  $1/r$  and it dominates for points near the origin. At low temperatures and for  $|\mu| < m$  the finite temperature effects are suppressed by the factor  $e^{-(m-|\mu|)/T}$ . In order to investigate the high temperature asymptotic, for the FC we have provided an alternative representation (2.39). At high temperatures, for points not too close to the origin, the FC is dominated by the Minkowskian part. The effects induced by the planar angle deficit and by the magnetic flux are suppressed by the factor  $e^{-2\pi r T \sin(\pi/q)}$  for  $q > 2$  and by the factor  $e^{-2\pi r T}$  for  $q < 2$ . The asymptotics at large distance from the cone apex are given by the expressions (2.45) and (2.46) for the cases  $q > 2$  and  $q < 2$ , respectively. The expression for the FC in the case  $|\mu| > m$  takes the form (2.47) with the upper and lower signs corresponding to  $\mu < -m$  and  $\mu > m$ . Now, the FC at zero temperature, given by (2.48), in addition to the vacuum part contains a contribution coming from the antiparticles ( $\mu < -m$ ) or particles ( $\mu > m$ ) filling the states with the energies  $m \leq E \leq |\mu|$ . For points near the apex, the zero temperature FC is dominated by the vacuum part whereas at large distances the contributions from particles or antiparticles dominate.

The contributions from particles and antiparticles to the charge density are given by (3.5) and are further transformed to (3.7) in the case  $|\mu| \leq m$ . For the total charge density one has the representation (3.11). In the case of a massless field with the zero chemical potential, as a consequence of the cancellation of the contributions from particles and antiparticles, the charge density vanishes. Similar to the FC, the charge density has indefinite parity with respect to the changes of the signs for  $\alpha_0$  and  $\mu$ . In the absence of the magnetic flux, the general expression is simplified to (3.13) and the charge density is an odd function of the chemical potential. The charge density for another special case of Minkowski bulk with magnetic flux is given by (3.14). The behavior of the thermal part in the charge density near the apex is similar to that for the FC. In this region the total charge density behaves as  $1/r$  and is dominated by the vacuum part. At large distances from the origin, the behavior of the topological part in the charge density is given by (3.21) and (3.22) for  $q > 2$  and  $q < 2$ , respectively. In this region one has an exponential suppression of the topological contributions. At low temperatures and for  $|\mu| < m$  the charge density is dominated by the vacuum part and the thermal effects are suppressed by the factor  $e^{-(m-|\mu|)/T}$ . At high temperatures, the main contribution comes from the Minkowskian part and the topological part behaves as  $e^{-2\pi r T \sin(\pi/q)}$  and  $e^{-2\pi r T}$  in the cases  $q > 2$

and  $q < 2$ , respectively. For the values of the chemical potential  $|\mu| > m$ , the expression for the charge density takes the form (3.23) with the upper and lower signs corresponding to  $\mu < -m$  and  $\mu > m$ . The contribution from the antiparticles or particles to zero temperature charge density is given by the second term in the right-hand side of (3.24). For a massless field this term survives only and the expression is simplified to (3.25).

The total charge induced by the planar angle deficit and by the magnetic flux is finite. For  $|\mu| \leq m$  it is given by the expression (3.30) with  $\Delta Q_0$  being the vacuum charge. In the case  $|\mu| > m$ , the charge at zero temperature receives an additional contribution from particles or antiparticles, depending on the sign of the chemical potential. This contribution is given by the second term in the right-hand side of (3.32). For a given sign of the chemical potential it is completely determined by the topological parameters of the model,  $q$  and  $\alpha_0$ .

In the problem under consideration, the only nonzero component of the current density is along the azimuthal direction. This component does not depend on the representation of the Clifford algebra. For the chemical potential in the region  $|\mu| \leq m$ , the corresponding expectation value is presented as (4.6). The current density has definite parity with respect to the reflections  $\alpha_0 \rightarrow -\alpha_0$  and  $\mu \rightarrow -\mu$ : it is an odd function of  $\alpha_0$  and an even function of  $\mu$ . In particular, the current density vanishes in the absence of the magnetic flux. For a massless field the general expression is simplified to (4.8) and in Minkowski bulk with the magnetic flux one has the expression (4.9). The thermal part of the physical component of the azimuthal current vanishes on the apex as  $r$  for  $2|\alpha_0| < 1 - 1/q$  and as  $r^{2\rho}$  in the case  $2|\alpha_0| > 1 - 1/q$ . The vacuum current diverges as  $1/r^2$  and dominates for points near the apex. At low temperatures and for  $|\mu| < m$  the finite temperature contribution is given by the second term in the right-hand side of (4.10) with the exponential suppression. At high temperatures the current density is suppressed by the factor  $e^{-2\pi r T \sin(\pi/q)}$  in the case  $q > 2$  and by  $e^{-2\pi r T}$  for  $q < 2$ . The large distance asymptotics are given by (4.15) and (4.16) in these two regions of  $q$ . For  $|\mu| > m$ , the expression for the current density has the form (4.17) and the zero temperature current density is given by (4.18). The latter consists two parts: the vacuum current and the current from particles or antiparticles filling the states with the energies  $m \leq E \leq |\mu|$ .

One can construct parity and time-reversal symmetric  $(2+1)$ -dimensional fermionic model by combining spinors realizing the two irreducible representations of the Clifford algebra. The corresponding Lagrangian density can be transformed to the form (5.1) with  $s = \pm 1$ . The expectation values in this model are obtained by using the formulas for separate representations (see (5.2)). The resulting FC is an even function of both the chemical potential and the parameter  $\alpha_0$ .

The charge density is an odd function of the chemical potential and an even function of  $\alpha_0$ . The vacuum charge density vanishes.

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## Appendix

In this appendix we derive the formulas used in Sect. 3 for the simplification of the expressions of the total charge. Let us consider the integral

$$\mathcal{P} = \int_0^\infty dx e^{-x} [\mathcal{I}(q, \alpha_0, x) - e^x/q], \quad (\text{A.1})$$

where  $\mathcal{I}(q, \alpha_0, x)$  is given by (2.15) with  $\alpha = \alpha_0$ . From the formula (2.17) it follows that the integral is convergent. We evaluate it in two different ways. Firstly, we use the expression (2.15). Inserting into (A.1), we note that the separate integrals with the first and second terms in the square brackets diverge. In order to have the right for separate integrations, we write the integral as  $\lim_{\lambda \rightarrow 1} \int_0^\infty dx e^{-\lambda x} [\dots]$ . For  $\lambda > 1$  both the integrals converge separately. By using the standard result for the integral with the modified Bessel function [71], after the summation over  $j$  and the limiting transition  $\lambda \rightarrow 1$  we get

$$\mathcal{P} = \frac{1 - q^2}{12q} + \alpha_0 (q\alpha_0 - 1). \quad (\text{A.2})$$

In the second approach for the evaluation of the integral (A.1) we use the representation (2.17). After the elementary integration over  $x$  one finds

$$\mathcal{P} = \frac{1}{q} \sum_{l=1}^p \frac{(-1)^l}{s_l^2} \cos(2\pi l(\alpha_0 - 1/2q)) - \frac{1}{\pi} \int_0^\infty dy \frac{f(q, \alpha_0, 2y)/\cosh^2 y}{\cosh(2qy) - \cos(q\pi)}. \quad (\text{A.3})$$

From (A.2) and (A.3) it follows that

$$\begin{aligned} & \sum_{l=1}^p \frac{(-1)^l}{s_l^2} \cos(2\pi l(\alpha_0 - 1/2q)) \\ & - \frac{q}{\pi} \int_0^\infty dy \frac{f(q, \alpha_0, 2y)/\cosh^2 y}{\cosh(2qy) - \cos(q\pi)} \\ & = \frac{1 - q^2}{12} + q\alpha_0 (q\alpha_0 - 1). \end{aligned} \quad (\text{A.4})$$

In the special case  $\alpha_0 = 0$  this gives:

$$\begin{aligned} & \sum_{l=1}^p \frac{(-1)^l}{s_l^2} c_l + \frac{2q}{\pi} \cos(q\pi/2) \int_0^\infty dy \\ & \times \frac{\sinh(qy) \sinh y \cosh^{-2} y}{\cosh(2qy) - \cos(q\pi)} = \frac{1 - q^2}{12}. \end{aligned} \quad (\text{A.5})$$

As another consequence of (A.4) one has

$$\begin{aligned} & \sum_{l=1}^p \frac{(-1)^l}{s_l} \sin(2\pi l\alpha_0) \\ & - \frac{q}{\pi} \int_0^\infty dy \frac{f_2(q, \alpha_0, y)/\cosh^2 y}{\cosh(2qy) - \cos(q\pi)} = -q\alpha_0. \end{aligned} \quad (\text{A.6})$$

Other relations are obtained from (A.4) by differentiation with respect to  $\alpha_0$ .

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